MATH-155 Lecture Notes
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Note: These notes are produced for didactical purposes only. They are supposed to help the students to identify the topics, which receive most stress in the course. They contain no original contribution of the Author, and should not be quoted. They should not be regarded as a substitute of the textbook from which many parts are drawn.

Lecture Notes:9 **Exponential Functions**

This section introduces the important class of functions called exponential functions. These functions are used extensively in modeling and solving a wide variety of real-world problem, including growth of money at compound interest; growth of population of people, animals, and bacteria; and radioactive decay.

In the last chapter, we introduce geometric growth. Recall that for any geometric growth we have the difference equation in the form of

$$a_{n+1}=ra_n \quad \textit{with} \quad a_0 \textit{ given}$$

and the corresponding functional equation is

$$a_n=r^n a_0 \quad \textit{with } p_0=100000$$

In the preceding chapter, in the example of the growth of population in the city of Homer was modeled, using p_n to stand for the population after n years, we wrote the following difference equation:

$$p_{n+1}=1.01p_n$$

says that the population at the end of each year is 1.01 times larger than the population at the end of preceding year. Equivalently, it says that the population is increasing 1% every year.

This difference equation leads to the functional equation for p_n as a function of n:

$$p_n=100000(1.01)^n$$

This is an example of an exponential function. As you know, n is referred to as an **exponent;** 1.01 is called **base**. In general, in an exponential function, the variable appears in the exponent attached to a constant base. Although this example defines n as whole number of years, the functional equation can be used even for fractional values of n. For the continuous time geometric model, the functional equation becomes

$$p(t)=100000(1.01)^t$$

Thus the model predicts 4.2 years after the start of first measurement, the population will be $100000(1.01)^{4.2}$.

To analyze the exponential functions in terms of their graphical and algebraic properties, we will use conventional variables x *and* y; where n will be replaced by x *and* p_n will be replaced by y. In our example, we can write our functional equation as follows:

$$y=100000(1.01)^x$$

This is one particular instance of an exponential function. The general form of any exponential function is $y=Ab^x$ *where* A and b stands for particular constants. In the example A=100000 and b=1.01.

Consider the following two functions:

$$f(x)=2^x \quad \textit{and} \quad g(x)=x^2$$

What is the difference between these two functions? Whether a variable appears as an exponent with a constant base or as a base with a constant exponent makes a big difference. The function *g* is a quadratic function, which we have already discussed. The function *f* is a new type of function called an exponential function. In general

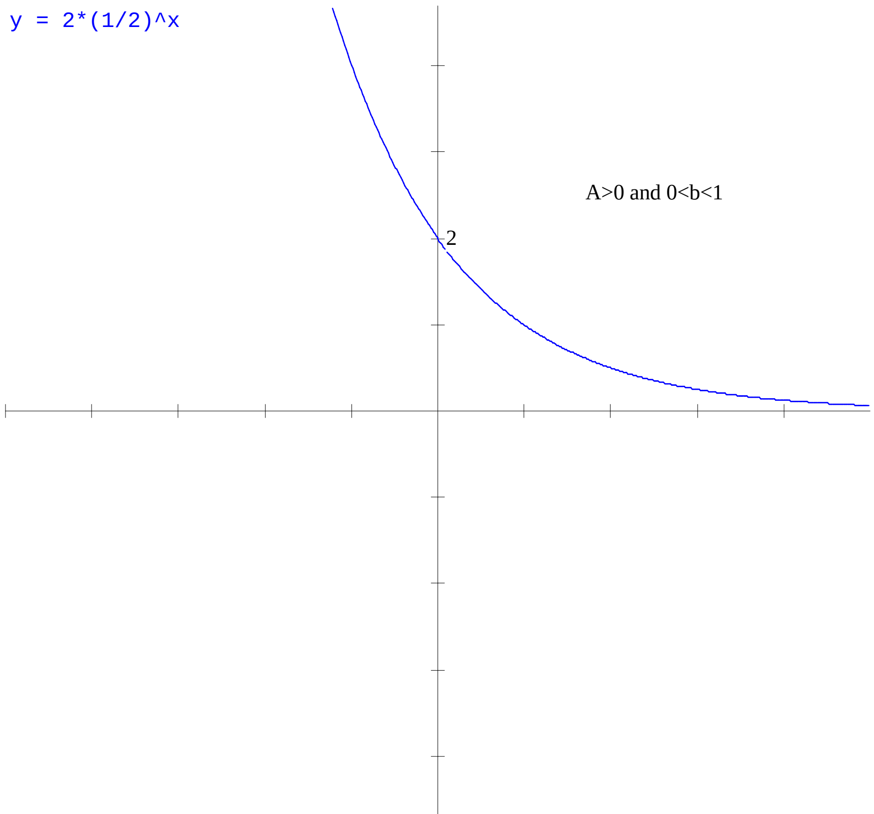
> The equation
> $y=Ab^x$ $\qquad b>0, b\neq 1$
> defines an exponential function for each different constant values of A and b. And the constant number b is called base.

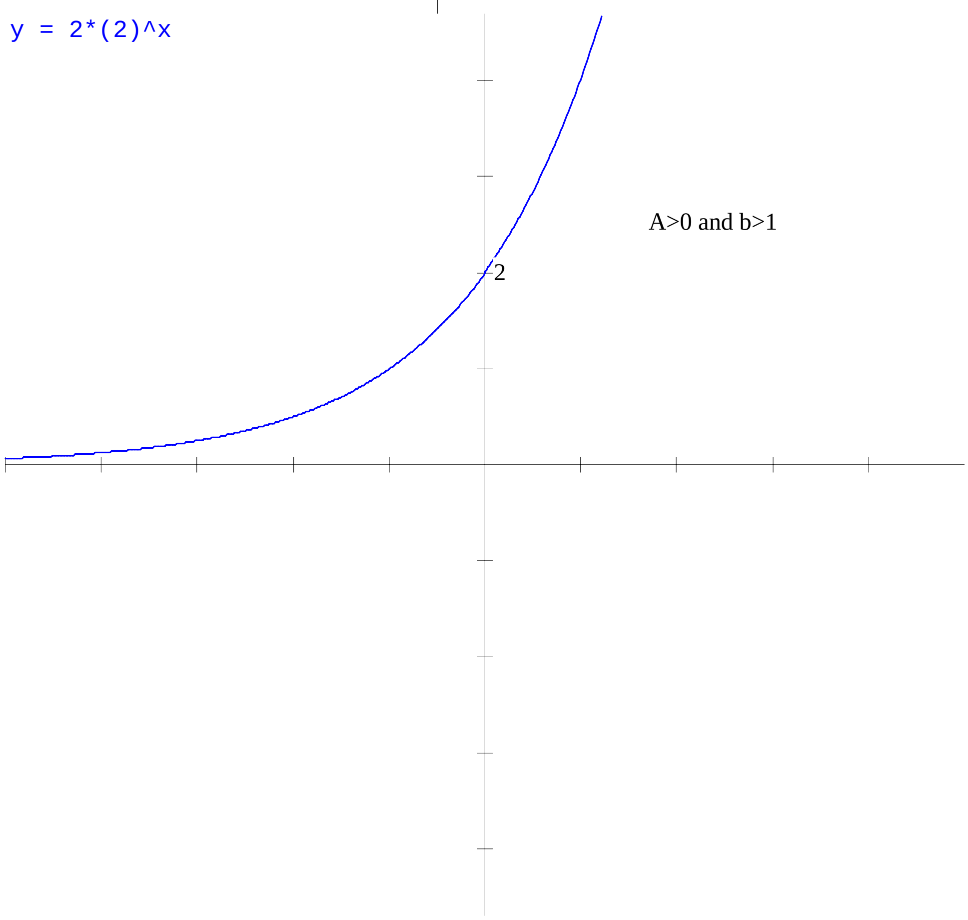
We require the base to be positive to avoid imaginary numbers. Also we exclude $b=1$ as a base, since when b=1, $b^x=1^x=1$ for any *x, which* implies when the base is 1 we have constant function. Therefore we exclude b=1 as a base.

Graphs

The general form any exponential function can be written as $y=Ab^x$. If we give different values to *A and b,* then we will have different exponential function. These exponential functions shares common characteristics in terms of their graphic according the value of A and b. Using our graphing software Winplot, we graph several exponential functions for different values of *A* and *b.* In these graph we see the following patterns for any exponential functions:

1) All graphs will pass through (0,A), i.e. the y-intercept of any exponential function is A. Since when $x=0$, $y=Ab^0$ and for any positive *b*, $b^0=1$. Therefore $y=A$
2) All graphs are continuous curves, with no holes or jump.
3) The *x*-axis is a horizontal asymptote. That is to say, there are no *x*-intercepts.
4) If $b>1$, *then* b^x increases as *x* increases. If $0<b<1$, then b^x *decreases as x* increases. That is to say, If $b>1$, then the curve is positively sloped. If 0<b<1,then the curve is negatively sloped. (See graph 1, 2, 3 and 4)
5) The value of *b* controls how steeply the curve rises or falls. As *b rises*, the curve becomes steeper if b>1. On the other hand as *b* rises, the curve becomes flatter if 0<b<1. (See graph 5)


y = 2*(1/2)^x
A>0 and 0<b<1
2


y = 2*(2)^x
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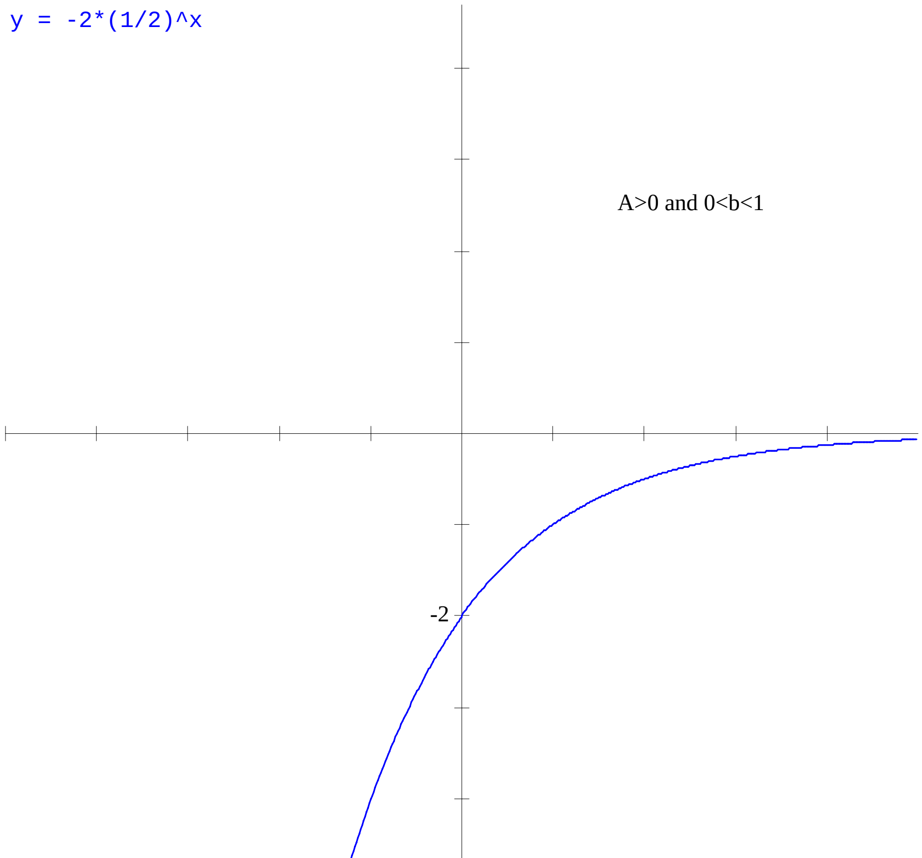
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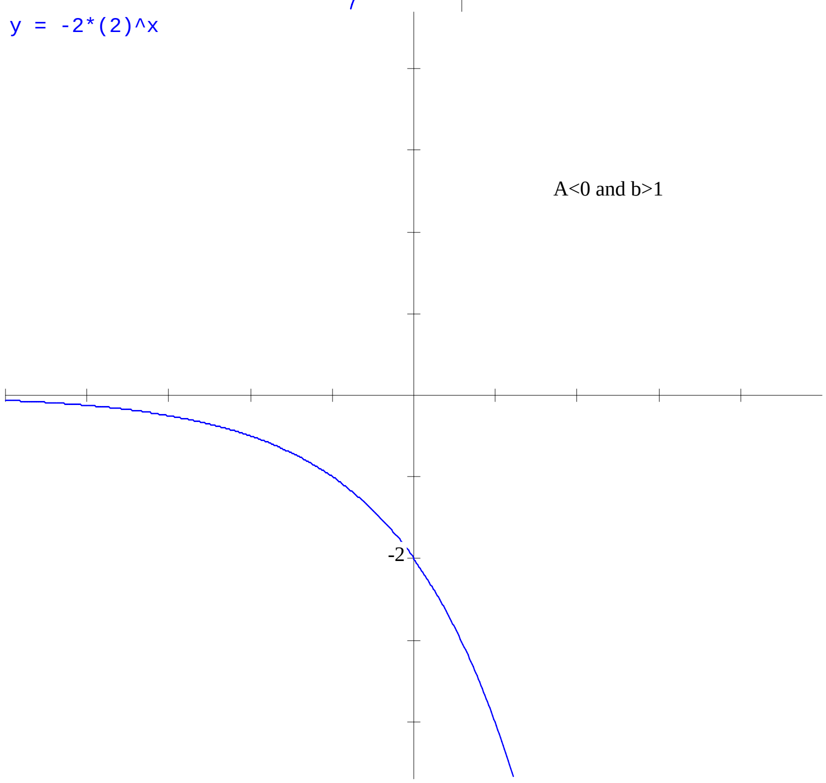
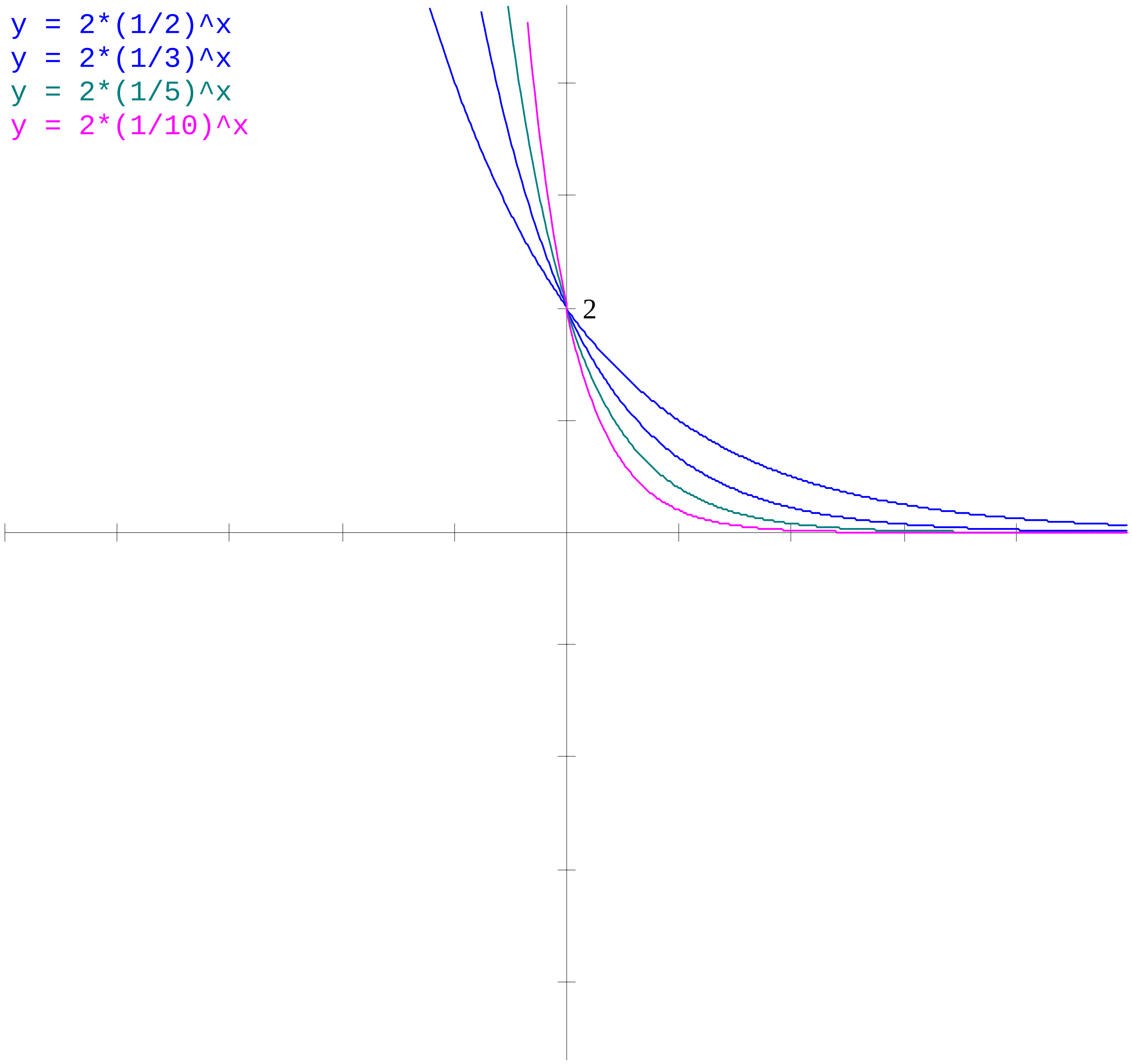
A>0 and 0<b<1
-2


y = -2*(2)^x
A<0 and b>1
-2



Algebraic properties of Exponential Functions

Consider our simplest population growth example where

$$p(t)=100000(1.01)^t$$

the symbols on the right hand side of the equal sign provide a recipe for computing the population. For example, if we want to know the population 4 years after the first measurement, the recipe says to raise 1.01 to the fourth power and then multiply the result by 100000. There are other possible forms for the recipe. One example is $101000(1.0201)^{0.5(t-1)}$. Although this looks quite different from $100000(1.01)^t$, it will produce the same result for any value of *t*. You can see some evidence of this by using a calculator to compute both $100000(1.01)^t$ and $101000(1.0201)^{0.5(t-1)}$ for several values of *t*.

t	0	1	2
$p(t)=100000(1.01)^t$	100000	101000	102010
$p(t)=101000(1.0201)^{0.5(t-1)}$	100000	101000	102010

Our discussion of algebraic properties of exponential functions will explain how to recognize when one form of an exponential function can be replaced by an equivalent form. As we shall see, there are situations in which one form easier to use than another.

The algebra of exponential functions depends on what are usually referred to as rules of exponents. As an example, let us consider what will result from multiplying 2^6 by 2^4. Remember that 2^6 is really 2*2*2*2*2*2. Similarly, 2^4=2*2*2*2. Therefore, 2^6*2^4=2*2*2*2*2*2*2*2*2*2, or 2^{10}. To express this in words, when we multiply a group of six twos with another group of four twos, the result is the same as multiplying together a group of ten twos. In symbols, this can be expressed in the form of the equation $2^4 2^6=2^{4+6}$. The same kind of reasoning can be applied for any base (in place of 2) and for any exponents (in place of 4 and 6). The result, expressed in the first rule of exponent

Rule 1: For *a* positive number, and *a≠1, and m and n are real numbers*

$$a^m a^n = a^{m+n}$$

Here is another rule of exponents. Raise 2 to the fifth power. Now take the result and raise that to the third power. What is the final result? In symbols, we wish to consider $(2^5)^3$. Now 2^5=2*2*2*2*2. When we raise that to the third power, we multiply together 3 identical copies of the same thing. That gives us (2*2*2*2*2)(2*2*2*2*2)(2*2*2*2*2). This is clearly the same as multiplying together 15 twos. That is , if we multiply together 5 groups of 5 twos each, the result is 15 twos, all multiplied. In symbols, $(2^5)^3=2^{5*3}$. Again the same reasoning applies for any base (not just 2) and any exponents (not just 3 and 5), so we have a second rule of exponents:

Rule 2: For *a* positive number, and *a≠1, and m and n are real numbers*

$$(a^n)^m = a^{nm}$$

Using the same logic, you should be able to come up the following two additional rules of exponents

Rule 3: For *a* positive number, and *a≠1, and m and n are real numbers*

$$\frac{a^n}{a^m} = a^{n-m}$$

Rule 4: For *a* positive number, and *a≠1, and m and n are real numbers*

$$a^n = a^m \quad \text{if and only if } n=m$$

The rules of exponents provide ways to modify how exponential functions are written.

In the last chapter, we saw the following pattern:

In a continuous model for geometric growth, suppose the variable *a* is a_0 at time $t=0$, and suppose that over *d* units of time there is a growth factor of *r*. *Then* $a(t)=a_0 r^{t/d}$

Consider the following example: A biologist observes the way a patch of mold grows in a glass dish, recording the size of the patch (in square inches) each month. She sees that the data follow a pattern: each month the patch 9 times larger than it was preceding month. On the first month the patch measured 8 square inches. The next month 9*8=72 square inches. The next month that 9*72=648 square inches and so on. Let a_t represent the size of patch *t* days after the first measurement. In terms of the statement in the box, $r=9$ $d=30$, *and* $a_0=8$. We can write our functional equation for the size of the patch as follows:

$$a(t)=8(9)^{t/30}$$

Now we try to write our functional equation in different form using the rules of exponent:

1) Using the fact that $9=3^2$, we can re-write our functional equation as follows:

$$a(t)=8(3^2)^{t/30}$$

Using the second rule of exponent

$$a(t)=8(3)^{2t/30}$$

$$a(t)=8(3)^{2t/30}$$

$$a(t)=8(3)^{t/15}$$

In this case a rule of exponent allows us to change an exponential equation in which the base 9 into one which the base 3. This implies that any base can be transformed to any other base. Of course, this allows us, if we wish, to express every exponential function using the base 10.

2) Recall our original functional equation:

$$a(t)=8(9)^{t/30}$$

This time we will re-write the exponent as follows: $\frac{t}{30}=\frac{1}{30}t$

Then;

$$a(t)=8(9)^{\frac{1}{30}t}$$

Using the second rule of exponent, we can express the exponential equation as follows:

$$a(t)=8(9^{\frac{1}{30}})^t$$

$$a(t)=8(1.076)^t$$

In this form, we can see that the patch grows by a factor of 1.076 each day.

3) As a final example, we will convert the equation into an exponential function with the base 10. Recall again our original functional equation:

$$a(t)=8(9)^{t/30}$$

and using the fact that $9=10^{.9542}$, we can re-write our functional equation as follows:

$$a(t)=8(10^{0.9542})^{t/30}$$

and using the second rule of exponent

$$a(t)=8(10)^{0.9542\ t/30}$$

$$a(t)=8(10)^{0.03181\ t}$$

The examples showed that the following equations are all equal:

$$a(t)=8(9)^{t/30}$$

$$a(t)=8(3)^{t/15}$$
$$a(t)=8(1.076)^{t}$$
$$a(t)=8(10)^{0.03181t}$$

The first equation allows us to see a glance that each month the patch 9 times larger than it was preceding month. Similarly, we see from the second equation that the size of the patch triples every 15 days, and from the third that the size of the patch grows by a factor of 1.076 each days. The last equation uses base 10, and would be convenient to use for numerical investigation on a calculator with a 10^x button.

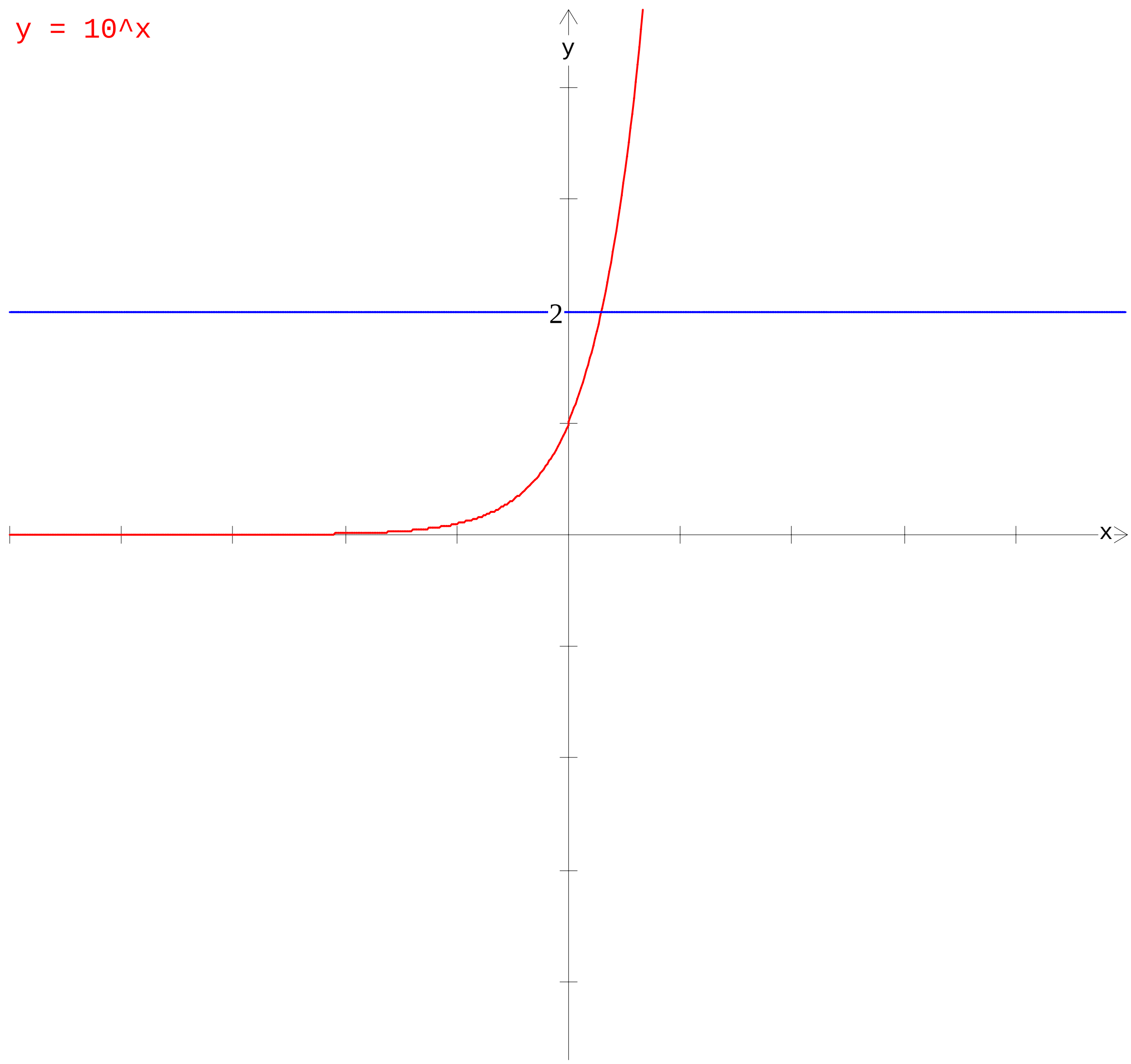
Solving Equations

An exponential equation is one for which a variable occurs in an exponent. The simplest form of such equation has a single exponential expression on one side of the equal sign, and a number on the other. A typical example is

$$10^x=2$$

As usual, we can approach a problem of this type graphically and numerically. In the following figure the graph of the equation $y=10^x$ is shown. To solve the equation $10^x=2$ graphically, we need to find a point on the curve where *y*-coordinate equals 2. A horizontal line shown in the figure crosses the *y* axis at 2. The point we seek where the horizontal line and the curve meet. Carefully draw a vertical line in the figure from the intersection point down to the *x*-axis. That tells you the *x* coordinate for the intersection point. You should come up with something very close to .3. That is the value of *x* we wanted, where $10^x=2$.

Now let's switch to a numerical method. Using a calculator, you will find that $10^{0.3}$ is little less than 2. On the other hand $10^{0.31}$ is a bit more than 2. So the solution to the equation $10^x=2$ must be between .3 and.31. This means that it must start out as 0.30. Using the calculator, compute 0.301, 0.302, 0.303, and so on. By using systematic trial and error in this fashion, you can eventually determine the solution to the equation. However, it is time consuming and sometimes very difficult to find exact solution to our equation. Therefore, both graphical and numerical method involve tedious process which might not give us exact solution.



Recall our simplest population example in which we had the following functional equation:

$$p(t)=100{,}000(1.01)^t$$

With this equation, we can predict the population at any time, simply by substituting that time for *t* and carrying out the computation indicated by the right hand side of the equation. This step is referred to as evaluating the function. However if we are asked the question when will the population reach 150000, this requires us to invert the function.

$$150000=100000(1.01)^t$$

Or equivalently, if we divide each side by 100000,

$$1.5=(1.01)^t$$

With the equation in this form we can apply graphical and numerical methods. That would be time-consuming process. To avoid this tedious process, we will introduce our theoretical model to solve this type of equation.

To solve these types of equations, we introduce the logarithmic functions. In theory there are different logarithmic functions for each base. For this section, we will concentrate on the base 10 logarithms, also referred to as common logarithm and base *e*, also referred to as natural logarithm.
Let us look at first the common logarithm. The log operation is the inverse function for the base 10 exponential functions. In the equation $10^x=y$, if you know x you compute *y*, that is evaluating the exponential function 10^x. Conversely, if you know *y* and wish to compute *x*, we use the log operation. That is inverting the equation. We write the solution as $x=log y$. So for the equation $10^x=7$, we express the solution as x=log 7 and use the calculator to find the numerical value of .845098. For the equation $10^x=2$, the solution is just $x=log2=0.301$. The log operation can be applied to any positive number. It cannot be applied to any negative number and 0.

How can logarithms actually be computed? One way is to compile a very complete table of values for the function 10^x. The table starts out like this.

x	10^x
0.001	1.0023052
0.002	1.0046158
…	…
0.300	1.99526
0.301	1.999
0.302	2.0044720
.845098	7
…	

Now you want to find out when 10^xequals 2 and 10^x equals 7. You look down the table until you find something as close to 2 and 7, respectively, as possible in the right-hand column. For the data in the table, .301 is the exponent that comes closest to giving the desired result of 2 and .845098 is the exponent that comes closest to giving the desired result of 7. You can use this idea to help you remember what a logarithm is. For example, think of log 2 as an abbreviation for

Locate the exponent **G**iving 2

Looked at in this way, a logarithm is an exponent; log 2 is the exponent you must put on 10 to produce 2 as a result.
So far, all of this discussion has dealt with a very special case of exponential equations with a base 10. However, the special case can be used to solve any exponential equation through a two-step process. First, we write each of the constants as a power of 10. This is done using logarithms. That will lead to a new equation, which we can easily solve.

To illustrate, here is how to solve the equation $1.5=(1.01)^x$. First we have to express 1.5 and 1.01 as powers of 10

$1.5=10^r$ and solve for r
$1.01=10^k$ and solve for k
The answer is
r=log 1.5=.1760912591
k=log 1.01=0.0043213738
Then re-writing our equation in the form of
$10^{0.1760912591}=(10^{0.043213738})^x$
Using the second rule of exponents, re-arrange the equation in the form of
$10^{0.1760912591}=(10)^{0.043213738\ x}$
Since the base in each side of equation is same, using the fourth rule of exponents we set
0.1760912591=0.043213738 x
then x=40.75
Does this solution give the answer to the original equation $1.5=(1.01)^x$?

$$1.5=(1.01)^{40.75}$$
$$1.5=1.5$$

But this is a very lengthy solution method. The solution to the equation $1.5=(1.01)^x$ can be written as

$$x=\frac{\log 1.5}{\log 1.01}=40.75$$

There is a simple pattern here. The solution to the equation $2^x=8$ is $x=\frac{\log 8}{\log 2}=3$. *This* pattern always hold, and it provides a simple way to solve exponential equations using the calculator with a log button.

The solution to the equation $b^x=c$ is given by $x=\log c/\log b$

Until now we use the common logarithm, which has the base 10. Actually for most application, the preferred base is an irrational number that is approximately 2.718281828, and we call this number *e*. In any scientific calculator, there is a key marked ln that computes the base *e* logarithm of any number. The *ln* label has an *l* for *log* and an *n* for natural, *and* the base *e* logarithm is often referred to as the *natural logarithm.*
The base *e* is an irrational number; it cannot be represented exactly by any finite decimal fraction. However, e can be approximated as closely as we like by evaluating the expression

$$\left(1+\frac{1}{x}\right)^x$$

for sufficiently large *x*. What happens to the value of the expression above as *x* increases without bound?

x	$\left(1+\frac{1}{x}\right)^x$
1	2
10	2.59374…
100	2.70481..
1000	2.71692..
10000	2.71814..
100000	2.71827..
1000000	2.71828

In fact as *x* increases without bound, the value of the $\left(1+\frac{1}{x}\right)^x$ approaches an irrational number that we call ***e***. The irrational number *e* to twelve decimal places is

e=2.718281828459

Exactly who discovered the constant *e* is still being debated. It is named after the great Swiss mathematician Leonhard Euler (1707-1783).

To analyze how to use natural logarithm to solve any exponential function, let us start with our simple population example. First, we will convert the function $p(t)=100{,}000(1.01)^t$ *to* an expression using base *e.* As usual, we begin by expressing 1.01 as a power of *e.* We need to find an exponent *r* so that $1.01=e^r$

By definition;

r=ln1.01=0.00995

then

$p(t)=100000\ (e^{0.00995})^t$

Using the second rule of exponents

$p(t)=100000\ e^{0.00995\ t}$

Whenever you are using the continuous time modeling, you should use the base e. If we are looking for the answer when the population will reach 150000, we will set *p(t)=150000.* Then the equation becomes

$150000=100000\ e^{0.00995\ t}$

$1.5=e^{0.00995\ t}$

The solution can be written as

0.00995 t=ln 1.5

0.00995 t=.405465

t=40.75

Or similarly using our original equation

$1.5=(1.01)^t$

The solution in terms of e base can be written as

t=ln1.5/ln1.01

t=40.75

This is the same solution we would have found using base 10 logarithms. To solve an exponential equation, you can use any base logarithms you please. To emphasize this fact, we repeat the previous comment using the natural logarithm.

The solution to the equation $b^x=c$ is given by $x=ln\ c/ln\ b$

Application 1: Exponential Growth

Cholera, an intestinal disease, is caused by a cholera bacterium that multiplies exponentially by cell division as given approximated by

$$N=N_0\ e^{1.386\ t}$$

Where N is the number of bacteria present after t hours and N_0 is the number of bacteria present at the start ($t=0$). If we start with 25 bacteria, how many bacteria (to the nearest unit) will be present

a) In 0.6 hours
b) In 3.5 hours

Our functional equation can be written as

$$N=25\ e^{1.386\ t}$$

Since $N_0=25$. To find out the number of bacteria 0.6 hours after the start, we just solve for N when t=0.6

$$N=25\ e^{1.386\ (0.6)}$$
$$N=25\ e^{0.8316}$$
$$N=25\ (2.297)$$
$$\boxed{N=57.425}$$

Similarly, to find out the number of bacteria 3.5 hours after the start, we just solve for N when t=3.5

$$N=25\ e^{1.386\ (3.5)}$$
$$N=25\ e^{4.851}$$
$$N=25\ (127.87)$$
$$N=3196.7$$

Instead of finding the number of bacteria at a certain time, suppose we are asked to find how many hours after the start the number of bacteria will reach 10000. Here we are given what the N is, we will find what the t is.

$$10000=25\ e^{1.386\ t}$$

dividing each side by 25 gives us

$$400=e^{1.386\ t}$$

To solve this equation, we can use several techniques which give us the same result.

1)

$$1.386t = \ln 400$$
$$1.386t = 5.99$$
$$t = 4.32$$

2) Using the second rule of exponent, we can re-write our equation as follows

$$400=(e^{1.386})^{t}$$
$$400=4^{t}$$

either using log operator or using ln operator gives us the solution for the t.

$$t=\frac{\log 400}{\log 4}=\frac{2.60206}{.60206}=4.32$$

Or

$$t=\frac{\ln 400}{\ln 4}=\frac{5.99}{.1.386}=4.32$$

Application 2: Compound Interest

Recall our compound interest formula

$$S=P(1+\frac{i}{m})^{m\,t}$$

where S is the compound amount, P is the principal, *i* is the annual interest rate, *m* is the number of conversion per year. Suppose that our individual wants to deposit $100 in her saving account for 2 years at an annual interest rate 8%. She wants to calculate her compounded amount after 2 years with different compounding frequency.

Here *P=100, i=0.08 and t=2, then*

Compounding Frequency	*m*	$S=100(1+\frac{0.08}{m})^{2m}$
Annually	*1*	$S=100(1+\frac{0.08}{1})^{2.1}=116.64$
Semi-annually	*2*	$S=100(1+\frac{0.08}{2})^{2.2}=116.9859$
Quarterly	*4*	$S=100(1+\frac{0.08}{4})^{2.4}=117.1659$
Monthly	*12*	$S=100(1+\frac{0.08}{12})^{2.12}=117.289$
Weekly	*52*	$S=100(1+\frac{0.08}{52})^{2.52}=117.3367$
Daily	*365*	$S=100(1+\frac{0.08}{365})^{2.365}=117.3490$
Hourly	*8760*	$S=100(1+\frac{0.08}{8760})^{2.8760}=117.3510$

Notice that the largest gain appears in going from annual to semi-annual compounding. Then the gains slow down as *m* increases. It appears that compounded amount, *S*, gets closer and closer to $117.35 as *m* gets larger and larger.

It can be shown that

$$P(1+\frac{i}{m})^{mt}$$

gets closer and closer to Pe^{it} as the number of compounding periods *m* gets larger and larger. The latter is referred to as the **continuous compounding interest formula,** a formula that is widely used in business, banking, and economics.

Continuous Compound Interest Formula

If a principal P is invested at an annual rate i (expressed as a decimal) compounded continuously, then the compound amount, S, in the account at the end of t years is given by

$$S=P\,e^{it}$$

If interest is compounded at the annual rate i, then the effective rate is given in terms of the exponential function with base e by $i_e = e^{i}-1$, where i_e represent the effective interest rate.

If our individual deposits her money in her saving account, which is compounded continuously, the she will get at the end of 2 years

$$S=100\,e^{0.08*2}$$

$$S=117.3510871$$

The effective interest rate is going to be $e^{0.08} - 1 = 1.08333 - 1 = 0.0833 = 8.33\%$.

Example: What amount will an account have after 5 years if $5,000 is invested at an annual rate of 10%?

a) Compounded monthly?
b) Compounded continuously?

Compute answers to the nearest cent.

Solution (a) Use the compound interest formula

$$S=P\left(1+\frac{i}{m}\right)^{mt}$$

with $P=5,000,\ i=0.10,\ m=12, \text{ and } t=5$

$$S=5000\left(1+\frac{0.10}{12}\right)^{12*5} \qquad \textit{Use a calculator}$$

$$S=8226.54$$

(b) Use the continuous compound interest formula

$$S=P\,e^{it}$$

with $P=5,000,\ i=0.10, \text{ and } t=5$

$$S=5,000\,e^{0.10*5} \qquad \textit{Use a calculator}$$

$$S=8243.61$$

The formulas for simple interest, compound interest, and continuous compound interest are summarized in the box at the top of the next page for convenient reference.

Interest Formulas

Simple Interest	$S=P(1+it)$
Compound Interest	$S=P\left(1+\frac{i}{m}\right)^{mt}$
Continuous Compound Interest	$S=Pe^{it}$

Example: How much you should you invest to have $8000 toward the purchase of a car in 3 years at annual interest rate 10%

a) Compounded quarterly?
b) Compounded continuously?

Solution (a) Use the compound interest formula

$$S=P(1+\frac{i}{m})^{mt}$$

with $S=8,000,\ i=0.10,\ m=4,\ and\ t=3$

$$8000=P(1+\frac{0.10}{4})^{4*3} \quad \textit{Use a calculator}$$

$$8000=P\ (1.345) \qquad \textit{divide each side by 1.345 to get P}$$

$$\boxed{P=5948}$$

(b) Use the continuous compound interest formula

$$S=P\ e^{it}$$

with $S=8,000,\ i=0.10,\ and\ t=3$

$$8000=P\ e^{0.10*3} \qquad \textit{Use a calculator}$$

$$8000=P(1.35) \qquad \textit{divide each side by 1.35 to get P}$$

$$\boxed{P=5926}$$

Example: How long will it take $100000 to grow $250000 if it is invested at 20%

(a) Compounded monthly?

(b) Compounded continuously?

Solution (a) Use the compound interest formula

$$S=P(1+\frac{i}{m})^{mt}$$

with $S=250,000,\ P=100,000,\ i=0.20,\ and\ \ m=12$

$$250,000=100,000(1+\frac{0.20}{12})^{12t}$$

$$2.5=(1.01667)^{12t}$$

$$12t=\frac{\ln 2.5}{\ln 1.01667}$$

$$12t=55.43$$

$$t=4.62 \text{ years}$$

(c) Use the continuous compound interest formula

$$S=P\ e^{it}$$

with $S=250,000,\ P=100000,\ and\ \ I=0.20$

$250000=100000\ e^{.20t}$

$2.5=e^{.20t}$

$0.20t=ln(2.5)$

$0.20t=.92$

$t=4.58$

Application 3: Exponential Decay

Cosmic ray bombardment of the atmosphere produces neutrons, which in turn react with nitrogen to produce radioactive carbon-14 (^{14}C). Radioactive ^{14}C enters all living tissues through carbon dioxide, which is first absorbed by plants. As long as a plant or animal is alive, ^{14}C is maintained in the living organism at a constant level. Once the organism dies, however, ^{14}C decays according to the equation

$$A=A_0 e^{-0.000124t}$$

where A is the amount present after t years and A_0 is the amount present at time t=0. If 500 milligrams of ^{14}C are present in a sample from a skull at the time of death, how many milligrams will be present in the sample in the sample in

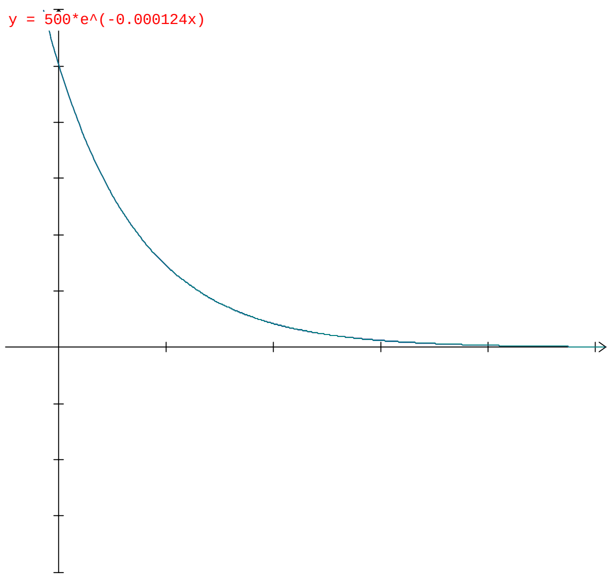
(a) 15,000 years?

(b) 45,000 years?

Solution:

Substituting $A_0=500$ in the decay equation, we have

$$A=500e^{-0.000124t}$$ (See the graph in the following figure)



(a) Solve for A *when* $t=15,000$

$$A=500e^{-0.000124*(15000)}$$ *Use a calculator*

$$A=77.84 \text{ milligrams}$$

(b) Solve for A when $t=45,000$

$$A=500e^{-0.000124*(45000)}$$ *Use a calculator*

$$A=1.89 \text{ milligrams}$$

Example: Refer to the exponential decay model in the above example. How many years after from the start ^{14}C level reduced to only 1 milligram.

Solution: Solve for t *when* $A=1$

$$1=500\, e^{-0.000124t}$$ *divide each side by 500*

$$0.002=e^{-0.000124t}$$

$$-0.000124t=\ln 0.002$$

$$-0.000124t=-6.21461$$

$$t=3107 \text{ years}$$

Terms and Concepts:

Effective interest rate

Base

Exponential function
Common logarithm
Natural logarithm
e
continuous compounding interest formula
continuous compounding