MATH-155 Lecture Notes
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Note: These notes are produced for didactical purposes only. They are supposed to help the students to identify the topics, which receive most stress in the course. They contain no original contribution of the Author, and should not be quoted. They should not be regarded as a substitute of the textbook from which many parts are drawn.

Lecture Notes: 10 Logarithmic Functions

Logarithmic functions are used in modeling and solving many types of problems. In this section we will study the logarithmic functions in detail in terms of their properties. Then we will discuss the logarithmic scales. Here the basic idea is to consider the logarithms of the data values, rather than using the original data directly. We will discuss the motivation for and advantages of using logarithmic scales. For example, the decibel scale is a logarithmic scale used to measure sound intensity, and the Richter scale is a logarithmic scale used to measure the strength of an earthquake.

Logarithmic Functions

As we stated preceding chapter, there are different logarithmic functions for each base. However we will discuss in this section only the base 10 and the base *e*. **Common logarithm** (also called **Briggsian logarithm)** is logarithm with base 10. **Natural logarithm** (also called **Napierian logarithm)** is logarithms with base *e*. Most calculators have a key labeled "LOG" and a key labeled "LN". The former represents a common (base 10) and the latter a natural (base *e*) logarithm. In fact, "log" and "ln" are both used extensively in mathematical literature, and whenever you see either used in this class (or any book) without a base indicated they would be interpreted as follows:

Logarithmic Notation

Common Logarithm: $\log x = \log_{10} x$

Natural Logarithm: $\ln x = \log_e x$

To begin, let us review the meaning of logarithms. Let us start with the following example:

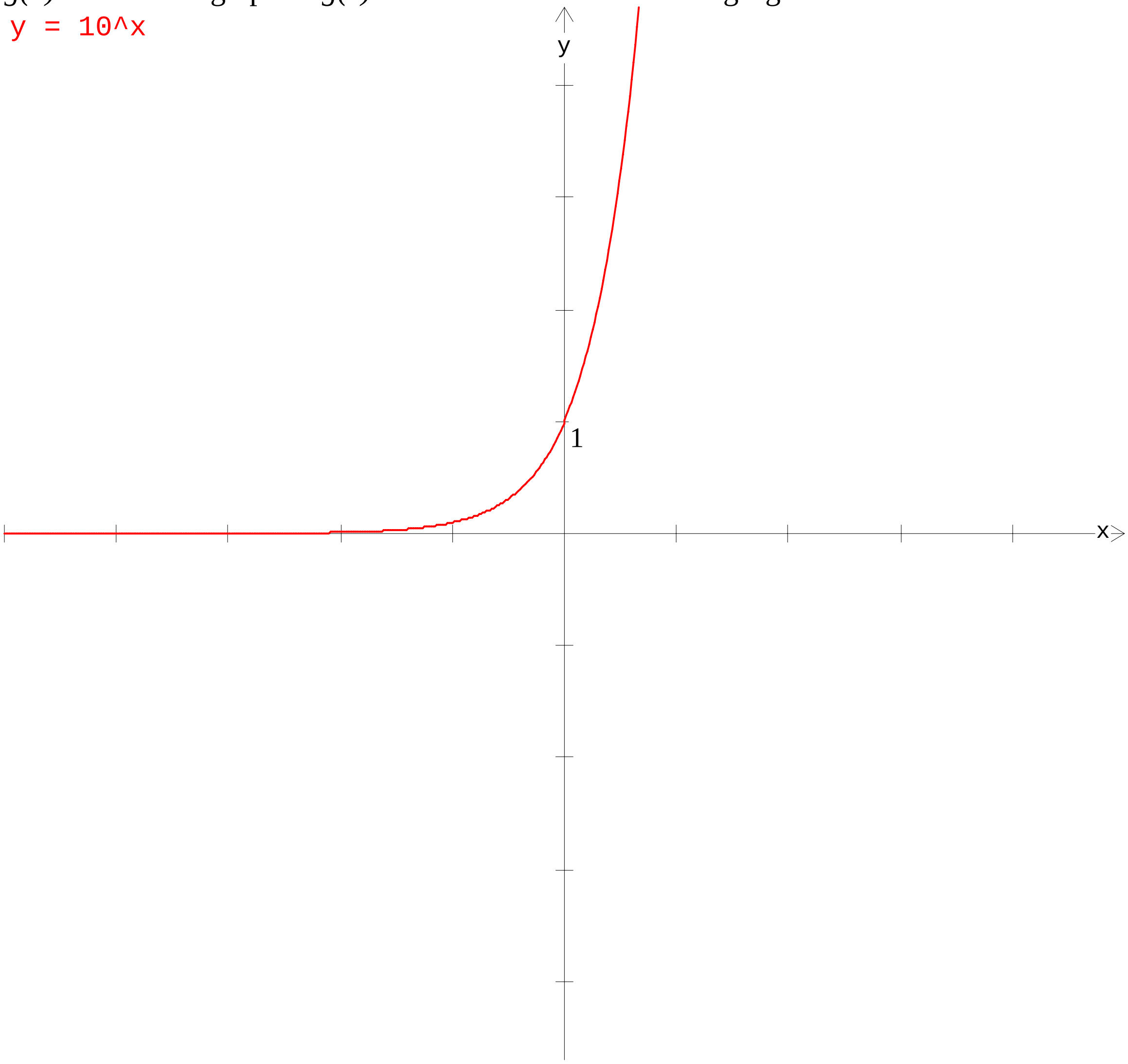
Example: (a) If $10^x=100$, what is *x*? (b) If $10^x=1000$, what is *x?* (c) If $10^x=500$, what is *x*?

Solution:

- ***(a)*** This problem asks the following question. To what power must we raise 10 to get 100? Since $10^2=(10)(10)=100$, the answer is 2.
- ***(b)*** To what power must we raise 10 to get 1000? Since $10^3=10*10*10=1000$, the answer is 3.
- ***(c)*** To what power must we raise 10 to get 500? The answer to this question is much harder than those in part (a) and part (b). Since $10^2=100$ and $10^3=1000$, the answer is somewhere between 2 and 3, but where? Let's try $10^{2.5}$. What does this mean? The easiest way to find $10^{2.5}$ is to use the 10^x key of a calculator. This gives $10^{2.5}=316.23$, which is too small. If we try $10^{2.6}$ using a calculator we obtain $10^{2.6}=398.11$, which is again too small. Trying another value, we obtain $10^{2.7}=501.19$, so 2.7 is close to our desired result. This trial and error method is not very satisfactory. Is there a better way? The best method is to use the log key

of a calculator. We then obtain log 500=2.69897 and the answer to (c) is x=2.69897. Thus log 500 is the power of 10 that equals 500.

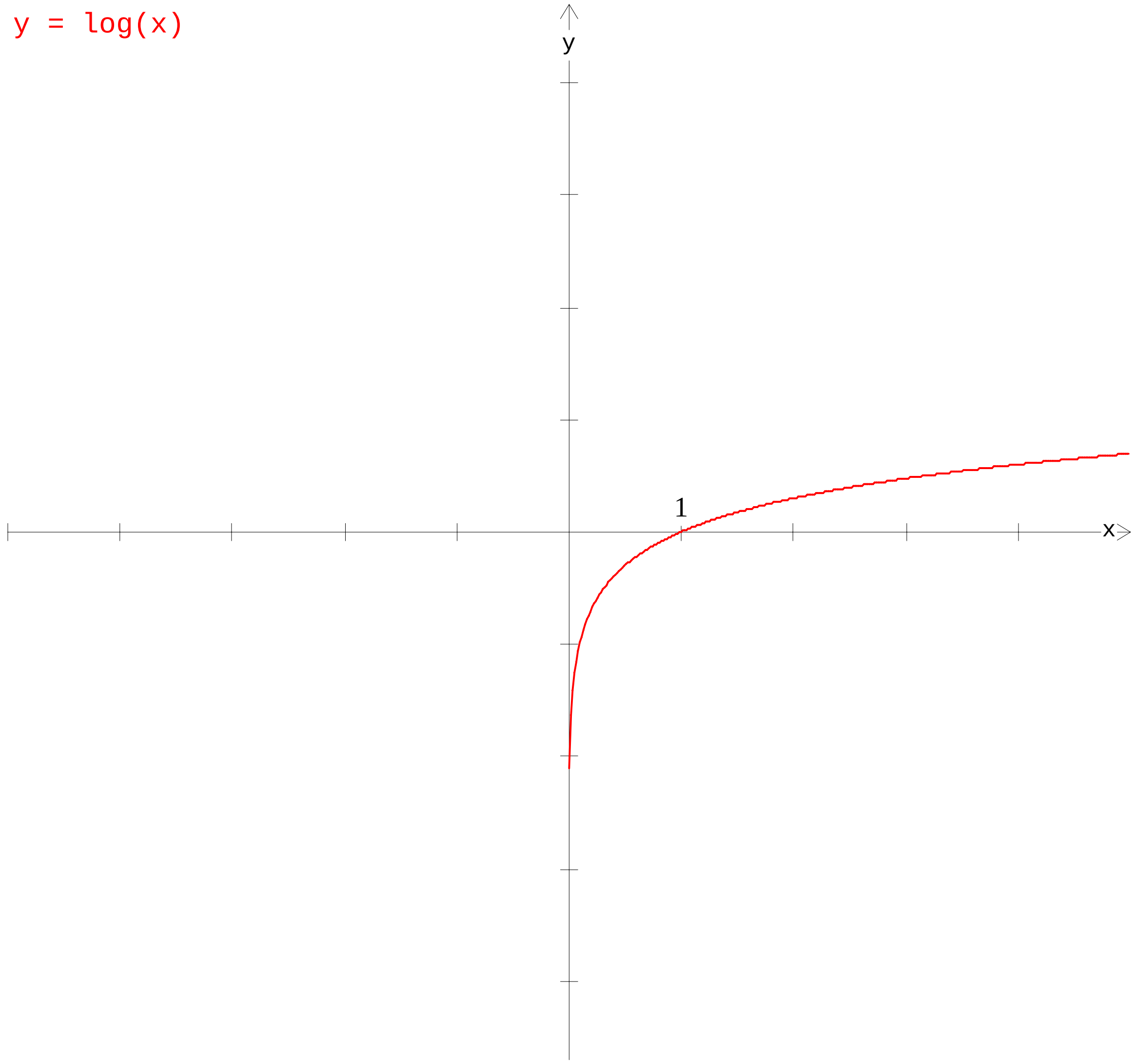
In this example we have introduced two functions, $f(x)=\log x$ and $g(x)=10^x$. Let's look at these two function more closely. You already know that $g(0)=10^0=1$, $g(1)=10$ $g(2)=100$, $g(3)=1000$. The graph of $g(x)$ is illustrated in the following figure.

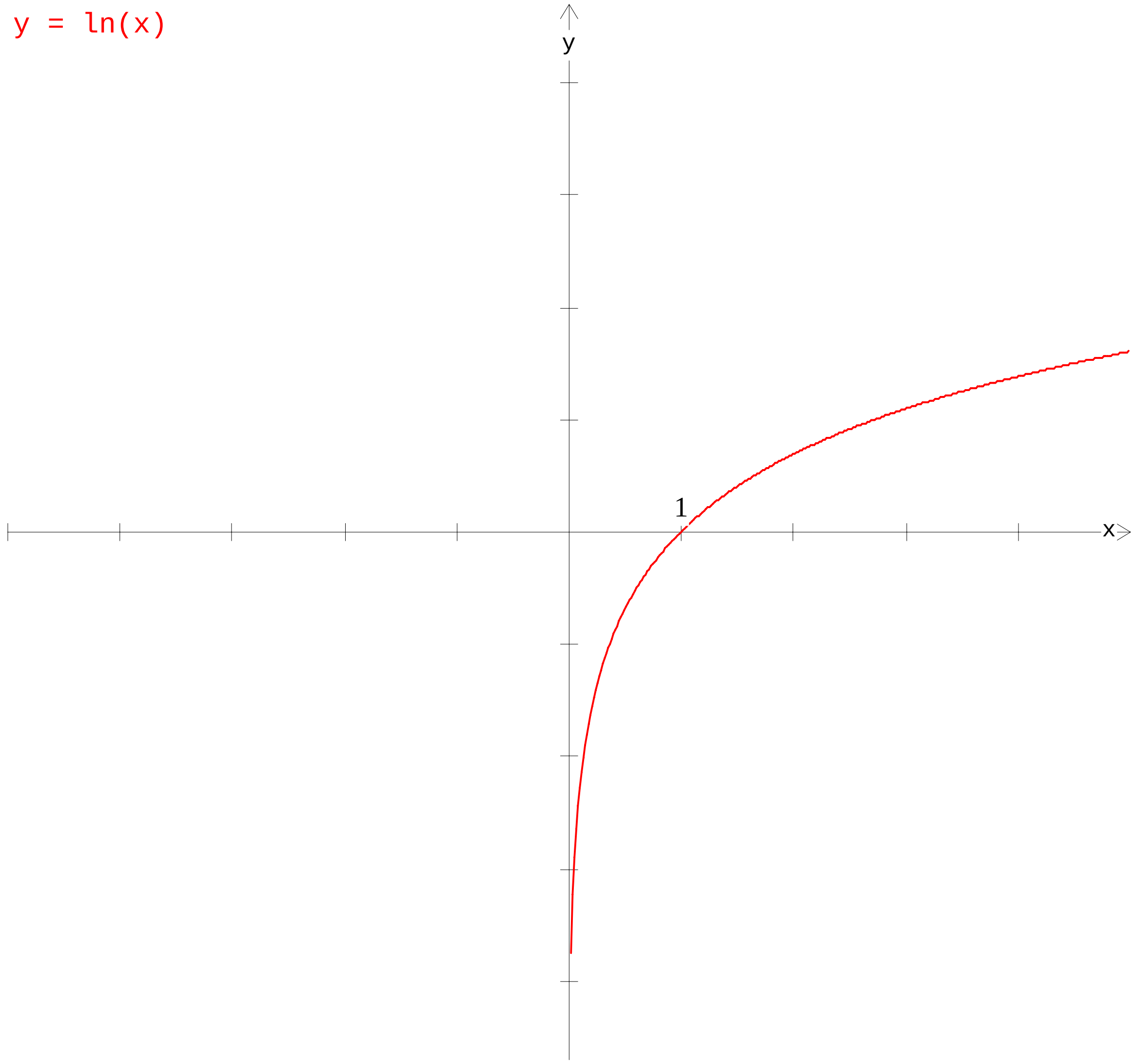


The function $f(x)=\log x$ is defined for all positive real numbers x and is the power of 10 that is needed to give x. More precisely, $y=\log x$ if $10^y=x$. We can obtain log x for certain values of x. For example,

log 1=0 since $10^0=1$ log 10=1 since $10^1=10$
log100=2 since $10^2=100$

If we want to graph $y=\log x$, we gave all possible positive values for x and resulting ordered pairs show us the shape of the graph.


y = log(x)
y
1
x



From the definitions of the functions $f(x)=log(x)$ *and* $g(x)= 10^x$ we see that if $f(x)=y$ *then* $g(y)=x$. More specifically, consider the following example, if we start with the exponential function g defined by

$y=10^x$

and interchange the variables; we obtain the inverse of g:

$x=10^y$

We call the inverse the **logarithmic function with base 10,** and write

$y=log\ x$ *if and only if* $x=10^y$.

Thus the statement $y=10^x$ has the same meaning as the statement $x=log\ y$. The statement $y=10^x$ is written in exponential form. The statement x=logy is written in logarithmic form.. Specifically, the statement $1000=10^3$ may be expressed in logarithmic form as 3=log1000

Example: Rewrite the statement $10^5=100000$ in logarithmic form.

5=log100000.

Studying the preceding examples, we note that a logarithm is an exponent of a number called the base. Thus, the statement

$$3=\log 1000$$

has the same meaning as the statement

$$10^3=1000.$$

Note that 3, the logarithm, is the exponent of 10, the base.

Here is our first rule of logarithm. Observe that 3=log 1000, then

$$10^{\log 1000}=1000$$

The same idea works for any positive number, therefore we can write our first rule of logarithm as follows:

For any positive number *a*

$$\boldsymbol{10^{\log a}=a}$$

Second rule is also straightforward from our example. Since $1000=10^3$, then

$$3=\log 10^3$$

The same idea works for any number. Therefore we can write our second rule as follows:

For any positive number *a*

$$\boldsymbol{\log 10^a=a}$$

The other rules of logarithms are closely related to the rules of exponents. We can state the other rules of exponent as follows

Rule 3: $\log a+\log b=\log(ab)$

Rule 4: $\log a-\log b=\log(a/b)$

Rule 5: $r\log a=\log a^r$

Rule 6: $\log 10=1$

Rule 7: $\log 1=0$

Rule 8: $\log a=\log b$ ***if and only if*** $a=b$

Since it is important to understand the meaning of these properties of logarithms, we will consider each in turn with an example.

Rule 3: This property states that the logarithm of a product of two numbers is equal to the sum of their logarithm.

$$\log(8*32)=\log 8+\log 32$$

$$\log 5+\log 6=\log(5*6)=\log 30$$

Rule 4: This property states that the logarithm of a quotient of two numbers is equal to the difference of their logarithm.

$$\log(5/9)=\log 5-\log 9$$

$$\log 6-\log 7=\log(6/7)$$

Rule 5: This property states that the logarithm of the r-th power of a number is equal to r times the logarithm of the number.

$$\log 4^6=6*\log 4$$

$$5\log 9=\log 9^5$$

Using Logarithmic Rules

Examples

1) Suppose you are working with the equation

$$y=100(1.02)^x$$

Applying the logarithm to each side we obtain a new equation

$$\log y=\log[100(1.02)^x]$$

$$\log y=\log 100+\log 1.02^x \qquad \text{Rule 3}$$

$$\log y=\log 10^2+x\log 1.02 \qquad \text{Rule 5}$$

$$\log y=2+x\log 1.02 \qquad \text{Rule 2}$$

$$\log y=.0086x+2 \qquad \text{since } \log 1.02=0.0086$$

2) $$\log\frac{wx}{yz}=\log wx-\log yz \qquad \textit{Rule}\quad 4$$

$$=\log w+\log x-\log y-\log z \qquad \textit{Rule}\quad 3$$

3) $$\log(wx)^{\frac{3}{5}}=\frac{3}{5}\log(wx) \qquad \textit{Rule}\quad 5$$

$$=\frac{3}{5}\log w+\frac{3}{5}\log x \qquad \textit{Rule}\quad 3$$

Solving Logarithmic Equations

Examples: Find *x so that*

1)

$\log x+\log(x+1)=\log 6$

$\log[x(x+1)]=\log 6$ *Rule 3*

$\log(x^2+x)=\log 6$

$x^2+x=6$ *Rule 8*

$x^2+x-6=0$ *Solve by factoring*

$(x+3)(x-2)=0$

$x=-3$ *and* $x=2$

We must exclude $x=-3$ since when $x=-3$, $x+1=-2$ *but* we don't have logarithm of negative numbers. Therefore the answer is only 2.

2)

$$\frac{3}{2}\log 4-\frac{2}{3}\log 8+\log 2=\log x$$

$$\log 4^{\frac{3}{2}}-\log 8^{\frac{2}{3}}+\log 2=\log x \quad Rule \quad 5$$

$$\log\left(\frac{4^{\frac{3}{2}}2}{8^{\frac{2}{3}}}\right)=\log x \quad Rule \quad 3 \quad and \quad 4$$

$$\log\left(\frac{\left(2^2\right)^{\frac{3}{2}}2}{\left(2^3\right)^{\frac{2}{3}}}\right)=\log x$$

$$\log\left(\frac{2^3 2}{2^2}\right)=\log x$$

$$\log 4=\log x$$

$$4=x \qquad Rule \quad 8$$

The rules presented here for base 10 or common logarithms have counterparts for any base logarithm. In particular, there is a variation of each rule for base *e* or natural logarithm.
We can summarize the rules for the natural logarithm as follows:

Rules of Natural Logarithm
For any *a,b*>0 and any real number *r*
Rule 1: $e^{\ln a}=a$
Rule 2: $\ln e^{a}=a$
Rule 3: $\ln a+\ln b=l(ab)$
Rule 4: $\ln a-\ln b=\ln(a/b)$
Rule 5: $r \ln a=\ln a^{r}$
Rule 6: $\log e=1$
Rule 7: $\ln 1=0$
Rule 8: $\ln a=\ln b$ ***if and only if*** $a=b$

Here is a final example using the rules of logarithm. Compare the logarithms of 345 and 3.45. These numbers have the same digits, differing only in the placement of the decimal point. That means that you can get one by multiplying the other by a power of 10: $345=10^2 * 3.45$. Now apply logs to both sides of the equation, and use the rule for adding

$$345=10^{2*}3.45$$
$$\log 345=\log(10^{2*}3.45)$$
$$\log 345=\log 10^2+\log 3.45$$
$$\log 345=2+\log 3.45$$

If you use the calculator to find the both the log of 345 and the log of 3.45, you will see the result above is correct. What is more, the log of 3.45 is not very large (at least smaller than 1) so the answer is pretty close to 2. This reveals a general rule for common logarithms: to approximate the common logarithm of any number, just count the number of digits in front of the decimal point. The logarithm will be less than the number of digits, but more than the next lower whole number. For example, since 456,568 has six digits in front of the decimal point, the log of 456,568 will be less than 6 but more than 5. This is because

$$456568=10^5*4.56568$$

$$\log 456568=\log(10^5*4.56568)$$

$$\log 456568=5+\log 4.56568$$

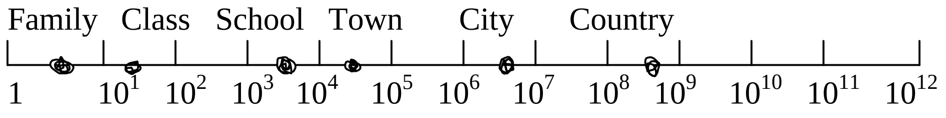
This relationship between the log of a number and the number of digits in front of the decimal point in closely related to the next topic of discussion, logarithmic scales.

Logarithmic Scales

There are many situations in which data cover so wide a range of values that it is difficult to comprehend the relationship in the data. To illustrate this idea, here is a rough comparison of several different population groups. The smallest group we consider is the family, which might have 4 members. Next, in a single class at school, there might be about 20 members. Continuing in this fashion, we can imagine the size of a school (4000), of a small town (20000), of a major city (5,000,000), and of the entire country (250,000,000).

Group	Family	Class	School	Town	City	Country
Size	4	20	4000	20000	5000000	250000000

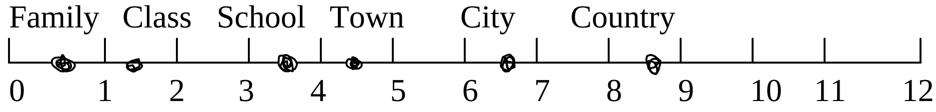
The numerical values in the table cover a wide range (from 4 to 250,000,000). If you try to compare these numbers putting on a number line, it is almost impossible to do that. In fact when you compare two numbers from the table, it is much more significant to look at the number of digits than the actual numbers involved. For example, the school size is in thousands (4 digits) whereas the city size is in millions (7 digits). The difference between the number of digits in these numbers is 3, indicating that one is about 1000 times the size of the other. When numbers are compared in this way, there is often reference to the idea of an order of magnitude. Quantities with different numbers of digit are described as having different orders of magnitude. That leads to a different way of making a number line. Make several marks evenly spaced out, and label these marks 1,10,100,1000,10000, and so on. On this type of line, the mark for a family size 4 would fall between the 1 and the 10; a class size of 20 is between 10 and 100, and so on. Because the different powers of 10 are equally spaced on this line, the position of each data point mainly reflects how many digits it has.



What does this have to do with logarithms? The number line in the above figure is called a logarithmic scale because it is equivalent to the graphing the logarithms of the population values on a normal line. As we saw at the end of the preceding section, the log of a number is closely related to the number of digits. The log of a 6-digit number 5 point something; the log of an 8-digit number is 7 point something, and so on. If we calculate the logarithms of the different groups population, we will get the following numbers:

Group	Family	Class	School	Town	City	Country
Log of Size	0.6	1.3	3.6	4.3	6.70	8.40

If we put these numbers on a normal number line, we will have



This is essentially the same number line we had except for the label below the number line. That is, plotting the logarithms of the raw data on a normal number line has the same effect as graphing the raw data on the new kind of number line used earlier, with equally spaced points indicating 1, 10, 100, 1000,and so on.

This new kinds of number lines, one with different powers of 10 evenly spaced, and the other with the logarithms of the data on the normal number line, are called a **logarithmic scale.**

Now take a look at the logarithms of different groups of population. Let's give a fancy name for this population data; Gump scale. The Gump scale value for a population figure is just the logarithms of the actual data. For example, in a town of 20000, the population measures 4.3 Gump scale, or in a city of 5000000, the population measures 6.7 Gump scale. Or the country measuring 8.4 Gump scale simply means that the logarithm of the population of that country is 8.

There are a few examples of this kind of plot that are part of everyday language. The Richter scale is for earthquake intensity; the decibel scale is for sound intensity; and the pH scale is for measuring the strength of acidity. Each of these is defined using logarithms, and each is also referred to as logarithmic scale. We will discuss the first two scales here, the third one, pH scale, is your responsibility to study from your book (page241-242).

Richter Scale

You frequently read in the newspaper that an earthquake has registered a certain amount on Richter scale. This scale was introduced by the American seismologist Charles R. Richter in the 1930s and is widely accepted measure of the strength of an earthquake. The Richter scale is a logarithmic scale for expressing the magnitude of an earthquake in terms of the energy dissipated in it. A reading of 1.5 indicates the smallest tremor can be felt, 4.5 results from an earthquake causing slight damage, and 8.5 indicates a devastating earthquake. The strongest earthquake of this century occurred in 1960 in southern Chile, with a Richter magnitude of 9.7. The strongest earthquake of this century in the United States was the famous San Francisco earthquake of 1906 with an 8.3 Richter magnitude.

If x is the energy dissipated by an earthquake, then the corresponding Richter scale reading is $y=log x$

The Richter scale is used because the energy x can get very large and the logarithm, as can be seen the graph of y=log x, brings large numbers down to a manageable levels.

Example: The 1906 San Francisco earthquake measured 8.3 on the Richter scale. What is the level of energy dissipated by this earthquake?

Solution: Let the energy of San Francisco earthquake be x_1 . We then log$x_1=8.3$. *Then the* amount of energy dissipated by this earthquake is $x_1=10^{8.3}$.

Example: The 1989 San Francisco earthquake measured 7.2 on the Richter scale, while an earthquake near Los Angeles in 1991 measured 6.3. How much stronger was the San Francisco earthquake than the Los Angeles earthquake?

Solution: Let the energies of San Francisco and Los Angeles earthquake be x_1 and x_2 , respectively. We have then log $x_1=7.2$ *and* log $x_2=6.3$. Hence

$$x_1 = 10^{\log x_1} = 10^{7.2}$$

$$x_2 = 10^{\log x_2} = 10^{6.3}$$

This gives

$$\frac{x_1}{x_2} = \frac{10^{7.2}}{10^{6.3}} = 10^{(7.2-6.3)} = 10^{0.9} = 7.94$$

We conclude that $x_1=7.94x_2$, so the San Francisco earthquake was 7.94 times stronger than the Los Angeles earthquake.

Decibels

Another important application of logarithmic scale is in computing decibels. A decibel is a unit for expressing the intensity of sound on a scale from zero for the least perceptible sound to about 150 for the severe ear damage. A decibel equals 1/10 of a bel, a unit named after the famous inventor, scientist Alexander Graham Bell. A quiet room registers about 40 decibels, while a noisy street registers about 80. The following table illustrates the noise level of various decibel readings.

Decibels	Noise Level
150	Severe ear damage
140	Jet take-off at close range
130	Threshold of pain
120	Amplified rock bond
110	Locomotive at close range
100	Circular saw
90	Dog barking
80	Vacuum Cleaner
70	Telephone bell
60	Conversation
50	Hummingbird
40	Soft wind
30	Quiet stream
20	Whispering
10	Pin hitting floor
0	Inaudible

If x is the intensity of a sound, then the corresponding decibel reading is

$$y=10 \log x$$

Example: If a jet airplane gives a decibel reading of 120 and a truck gives a reading of 90, how much louder is the plane than the truck?

Solution: Let the sound intensities of the plane and truck be x_1 *and* x_2, respectively. Since 10 log$x_1=120$ and 10 log$x_2=90$, *we have*

$$\log x_1=12 \text{ and } \log x_2=9$$

Using the rules of logarithm, we can find the sound intensities as follows

$$x_1 = 10^{\log x_1} = 10^{12}$$

$$x_2 = 10^{\log x_2} = 10^{9}$$

Hence,

$$\frac{x_1}{x_2} = \frac{10^{12}}{10^{9}} = 10^{3}$$

We conclude that $x_1=10^3x_2$, so the plane is 1000 times louder than the truck.

This completes the discussion of logarithmic scales. The main ideas of these discussions are:

- Logarithmic scales are used when the actual data cover a vast numerical range.
- Logarithmic scales are defined using the logarithms of the data, rather than the data themselves.

Terms and Concepts
Common logarithm
Natural logarithm
Base *e*
Richter scale
Decibel scale
Logarithmic scale