MATH-155 Lecture Notes

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Note: These notes are produced for didactical purposes only. They are supposed to help the students to identify the topics, which receive most stress in the course. They contain no original contribution of the Author, and should not be quoted. They should not be regarded as a substitute of the textbook from which many parts are drawn.

Week I Introduction

Data resulting from the measurement of variables may come from any number of sources and in a variety of forms. Data, for example, which describes the movement of a variable over time are called time-series data, and may be daily, weekly, monthly, quarterly, or annual. Suppose we are interested in the relationship between two variables X and Y (Let's assume X is the year and Y is the level of real GDP in the US). In order to describe a relationship between these two variables, we need a set of observation for each variable and mathematical form of the relationship. The set of observations is called a sample data. We will be concerned primarily with the case in which the relationship between X and Y is linear, i.e. described by a straight line.[i]

If we look at real life, on the other hand, such as economic events as a whole, it looks almost chaotic and impenetrable, and this might be first insight of every social scientist. There is no exact and one economic model based on mathematical model to solve the problems he faces. Answer to one problem at one point may not be the answer for the same problem at another point in time. It might possible that if you insist using this answer to your social or economic problems, it will create disaster rather than solving your problems.

Keynes (1936) argued " Practical men, who believe themselves to be quite exempt from any intellectual influences, are usually the slaves of some defunct economist. Madman in authority, who hear voices in the air, are distilling their frenzy from some academic scribbler of a few years back. I am sure that the power of vested interest is vastly exaggerated compared with the gradual encroachment of ideas. Not, indeed, immediately, but after a certain interval, for in the field of economic and political philosophy there are not many who are influenced by new theories after they are twenty-five or thirty years of age, so that the ideas which civil servants and politicians and even agitators apply to current events are not likely to be the newest. But soon or late, it is ideas, not vested interests, which are dangerous for good or evil."[ii]

It is the mathematical language, notation and techniques, which helps to formalize ideas or theories. Social scientist develops a very simplified image of reality; i.e***. model,*** to analyze and explain the behaviors that are most important to us and omit details that are relatively unimportant. A model should depict those parts of reality that are considered relevant to a certain problem. The model is not merely allowed to abstract from reality, on the contrary, it is obliged to. Abstraction and simplification are the inherent functions of a model. A good model yields some interpretation, predictions and forecasting of those phenomena.

Typically, a model includes the following elements. First there are definitions and variables. Then we have some axioms; i.e. unexplained and unproved proposition that are

assumed true in the model. Finally there are empirical laws gathered from some systematic observation. The results are deduced. They are called theorems, implications or conclusions. In this sense, models are means of developing theories.

By a method, we mean a special procedure for obtaining scientific results. Hence a method is a technique of analysis or a way of looking at things.

In this class, we are going to look at some methods that are used in applying mathematics to real world problems. With these methods we try to obtain a model, which partially explain the problems and draw some prediction for the future.

Consider the following example. The table shows the levels of real gross domestic product in US between 1990 and 1996. Real GDP is one of the important macro data in national income accounting, which measures the value of final goods and services at fixed prices for a one-year period. For example, in 1990, US real GDP was 6 trillion 136 billion 300 million dollar.

Year	Real GDP (Billions of 1992 Dollars)
1990	6136.3
1991	6079.4
1992	6244.4
1993	6386.1
1994	6608.4
1995	6742.2
1996	6906.8

Using this table we can answer many question, like

a) How fast is the real GDP level growing?
b) When did it first reach 6 trillion dollar?
c) When is it expected to reach 10 trillion dollar?
d) What will it be 10 years from 1996?

To answer these questions we are going to use three types of methods: ***numerical, graphical and theoretical.***

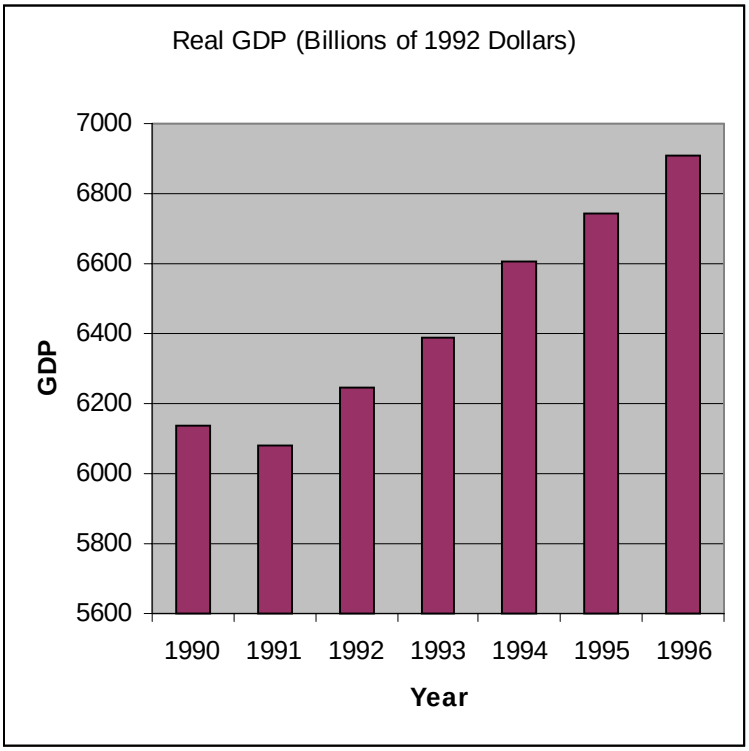
NUMERICAL METHOD

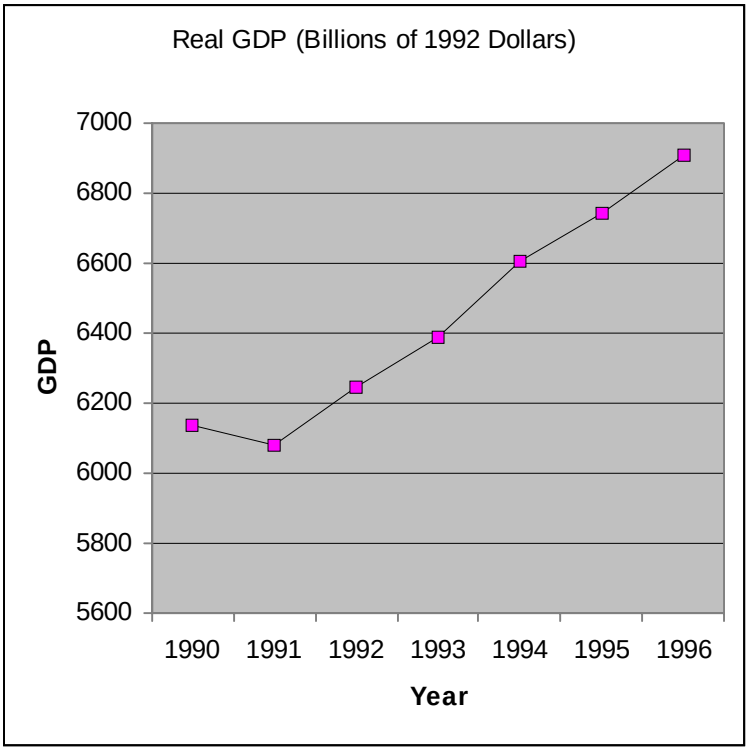
Numerical methods involve the actual data that appear in the table. If we look at the table, we see immediately an answer for how fast the GDP growing. For example, in 1996 real GDP growth was %2.44.

$$Growht\ \ Rate\ of\ \ GDP_{1996} = \frac{GDP_{1996} - GDP_{1995}}{GDP_{1995}} * 100$$

Graphical Method

Graphical methods use pictorial or spatial representations to help us understand and communicate relationship. It is common to use bar graphs or line graphs to depict data of the type shown in the table.


Real GDP (Billions of 1992 Dollars)
7000
6800
6600
6400
6200
6000
5800
5600
GDP
1990
1991
1992
1993
1994
1995
1996
Year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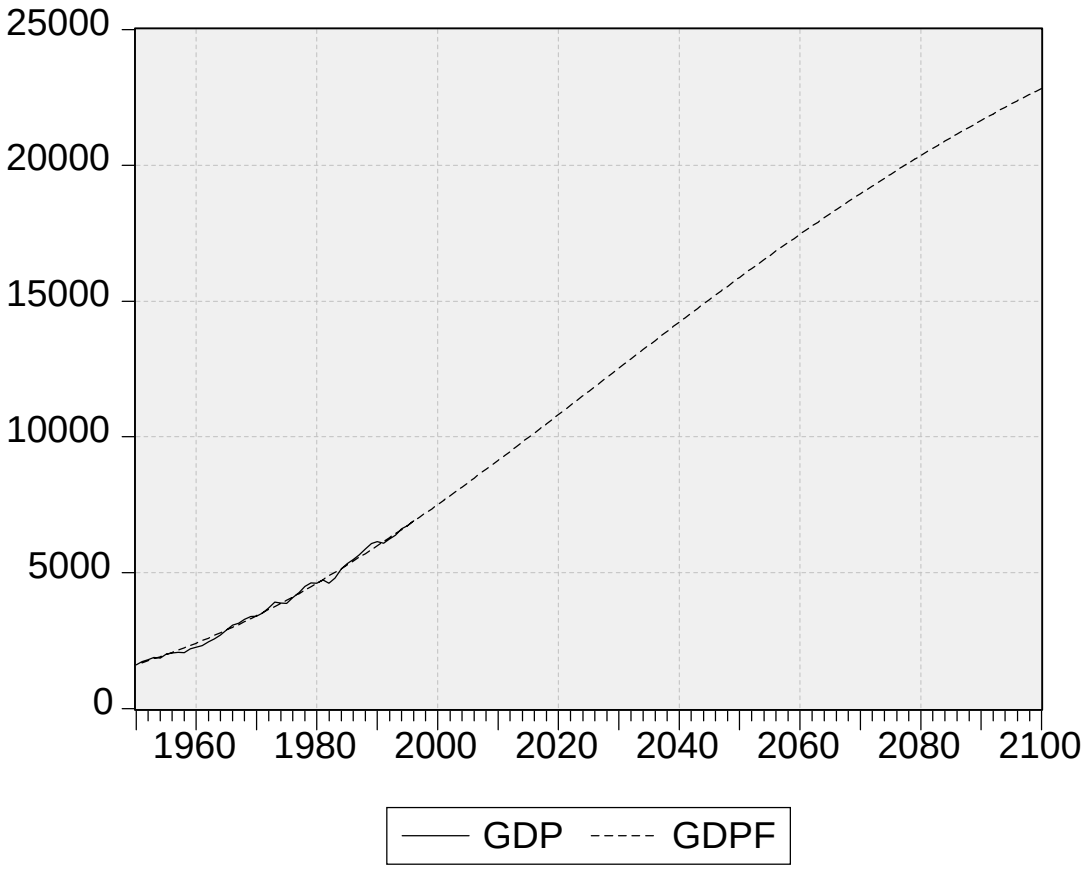
Year



Theoretical Methods

It employs mathematical tools for expressing and manipulating relationships or patterns. These tools typically involves some variables, in our example real GDP and year, and algebra. We try to find a relationship between these variables, specifically we ask the question: How these variables move together? For example using least square estimation method for the period between 1990 and 1996; we find

$$\text{Real GDP Level} = 142.8964\,\text{Year} - 278349$$

Question: When the real income will reach 10 trillion dollars?

If we look at the graphic drawn for estimated real GDP, it shows us real GDP will reach 10 trillion dollars around 2018. We are not sure whether real GDP will reach 10 trillion dollars in 2018 or not, but we have rough estimate for real GDP level in the US from the graphic. In the graphical method, to get an estimate of real GDP level in the future, all we need to do is to enlarge the line graph so that real GDP level, or vertical axis, goes up to 10 trillion dollars.

Numerical method involves in this situation using the estimation equation along with graphic. Using this graphic and estimation equation, let's look at real GDP level around 2018.

Real GDP Level$_{2018}$= 142.8964(2018)-278349*
Real GDP Level$_{2018}$=10015.93

This is approximately correct, but still we do not have exact number 10000. This number is a little too high. It must be that US economy, given current conditions, reach that level of income earlier than 2018. Let us try 2017?
Real GDP Level$_{2017}$= 142.8964*(2017)-278349
Real GDP Level$_{2017}$=9873.0388

We can conclude that real GDP reach 10 trillion dollars between 2017 and 2018, and it is close to 2018.

On the other hand, theoretical model gives us the opportunity that we can calculate exact timing. Here we want to find out what year US economy will produce 10 trillion dollars of real output. In our equation, put 10000 in place of real GDP level, and leave the year as an unknown. To find the year follow these steps:

$$\text{Real GDP Level} = 142.8964\,\text{Year} - 278349$$

$$10000 = 142.8964\,\text{Year} - 278349$$

$$10000+278349=142.8964\,\text{Year}$$

$$288349=142.8964\,\text{Year}$$

$$\text{Year} = \frac{288349}{142.8964}$$

Then,
Year=2017.89

Question: Use graphical, numerical and theoretical model to estimate the year when the real GDP level will reach 5 trillion dollars. Repeat this exercise for 7.5 trillion dollars.

From these experiments, three observations arise:

1. These method complement each other, they are used together to understand the data and to predict the future.
2. Theoretical model has greater power and generality than the other methods.
3. All three approaches to predict the future suffer from a common limitation. While graphical method depends on the assumption that the straight line is an accurate portrayal of the data, the other two methods assume the estimation equation is correct.

Mathematical Models
Mathematical models are always subject to improvement. New data collections can change our estimation equation and our purpose is to find best mathematical model that can explain problem we are trying to understand. Every new model might give us different predictions. As we collect new data, we will test our model to see the model's ability to predict future.

Data can be daily, weekly, monthly, quarterly or annual. That is to say, time is taken to be a *discrete variable*, so that the time variable is allowed to take only integer values. Looking at regularly spaced data, we will try to find patterns that follow simple mathematical description.

What we are trying here is to estimate future by using past data. Patterns of this type are described by mathematical equations called *difference equations.* In our example, the difference equation for the GDP level in US says, in essence, the GDP level this year will be previous year's GDP level plus 142.8964.

$$GDP_t = GDP_{t-1} + 142.8964$$

In this example, we use data from 1990 to 1996. That is to say, our starting period is 1990 and the level of real GDP in 1990 is 6136.3. To find the estimated value of real GDP in 1991, we just add 142.8964 to the 1990 level of GDP.

$$GDP_{1991} = GDP_{1990} + 142.8964$$
$$GDP_{1991} = 6136.3 + 142.8964 = 6279.1964$$
$$GDP_{1992} = 6279.1964 + 142.8964 = 6422.0928$$
$$GDP_{1993} = 6422.0928 + 142.8964 = 6564.9892$$

The difference equation can be used to calculate level of GDP for any year since each year level of GDP is closely related to previous year's GDP. But this way of calculation becomes quite tedious if we want to find the level of GDP in the long run, say 2050.

Instead, sometimes it is useful to use *functional equations* to calculate future values of GDP, or any other variable. The equation

$$\text{Real GDP Level} = 142.8964\,Year - 278349$$

is an example of such functional equation. Here real GDP level is a function of year. This means that, as soon as you put year in which you want to find level of GDP, all you need to do is to replace that year in our equation. For example, estimated level of real GDP in 2000 will be

$$\text{Real GDP Level} = 142.8964 * 2000 - 278349 = 7443.8$$

If the variable depends on its own past values, we call such equations as difference equations. On the other hand, if the variable depends on other variable, we call such equation as functional equation. Here, in the first case, real GDP level depends on previous period real GDP level, therefore it represents a difference equation. In the second case, real GDP level depends on year, therefore it represents a functional equation.

Terms and Concepts

Method	*Model*
Difference Equations	*Functional Equations*
Numerical Method	*Graphical Method*
Theoretical Methods	*Real GDP*
Growth Rate of GDP	

Problems for Review

1. Write a short essay, less than two pages, in which you summarize the materials in chapter 1. Specifically, you should mention the numerical approach, the graphical approach, and the theoretical approach and how they are used to analyze a collection of data. What are some advantages and disadvantages of each approach? How are difference equations and functional equations used? Think of a collection of data, different from the examples given in the class, that you might find in a field you are interested in and discuss the types of questions you could answer by analyzing this data using the methods of chapter 1.
2. Instead of using only data from 1990 to 1996, if we use data from 1950 to 1996, we will get estimation equation as follows

$$\text{Real GDP Level} = 117.075\ \textit{Year} - 227062$$

   a) Using this equation, find the years when the level of real GDP will reach 5, 7.5 and 10 trillion dollars. Compare your results with the answers which we obtained from using estimation equation for data from 1990 to 1996.
   b) Write difference equation and functional equation for these data period.

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[i] Pindyck R.S and D. Rubinfeld "Econometric Models and Economic Forecast" New York: McGraw-Hill Inc, 3rd ed., 1991

[ii] Keynes J.M. "General Theory of Employment, Interest and Money'' London: Macmillan, 1936