MARKET RISK MANAGEMENT: MEASUREMENT: MONTE CARLO SIMULATION

- Real Data
- Monte Carlo Simulation

REAL DATA

Returns

Distribution

Stylized Facts

ASSET RETURNS

- *Returns Discrete Compounding*

$$r_t = (S_t - S_{t-1})/S_{t-1} \qquad (3.1)$$

- *Returns Continuous Compounding*

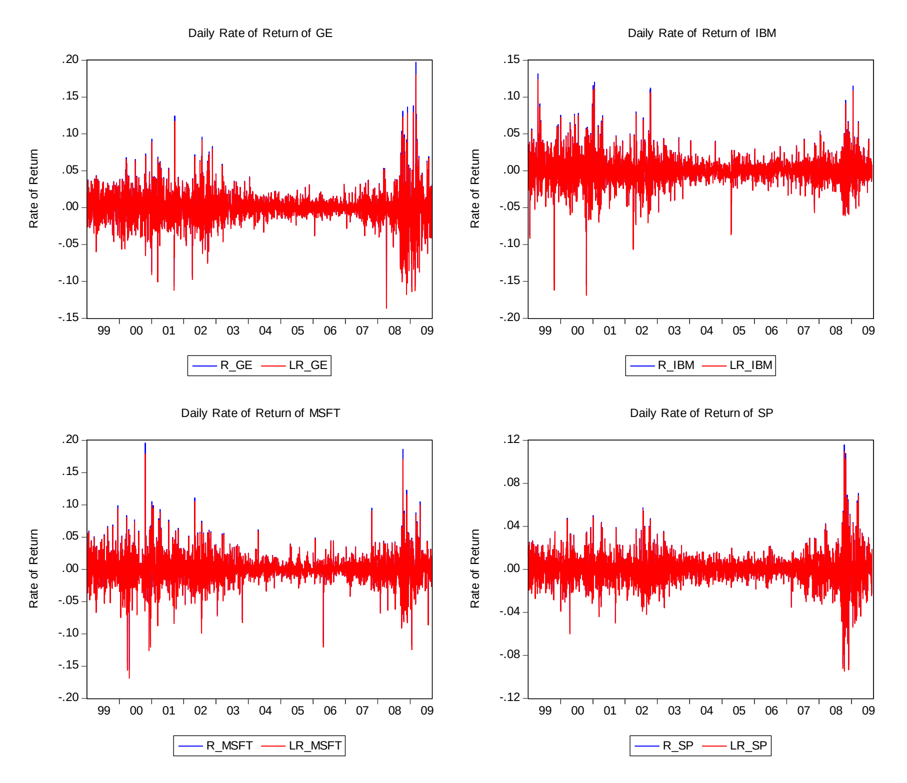
$$R_t = ln[S_t \,/\, S_{t-1}] \qquad (3.2)$$

- Equation (3.1) and (3.2) are related by

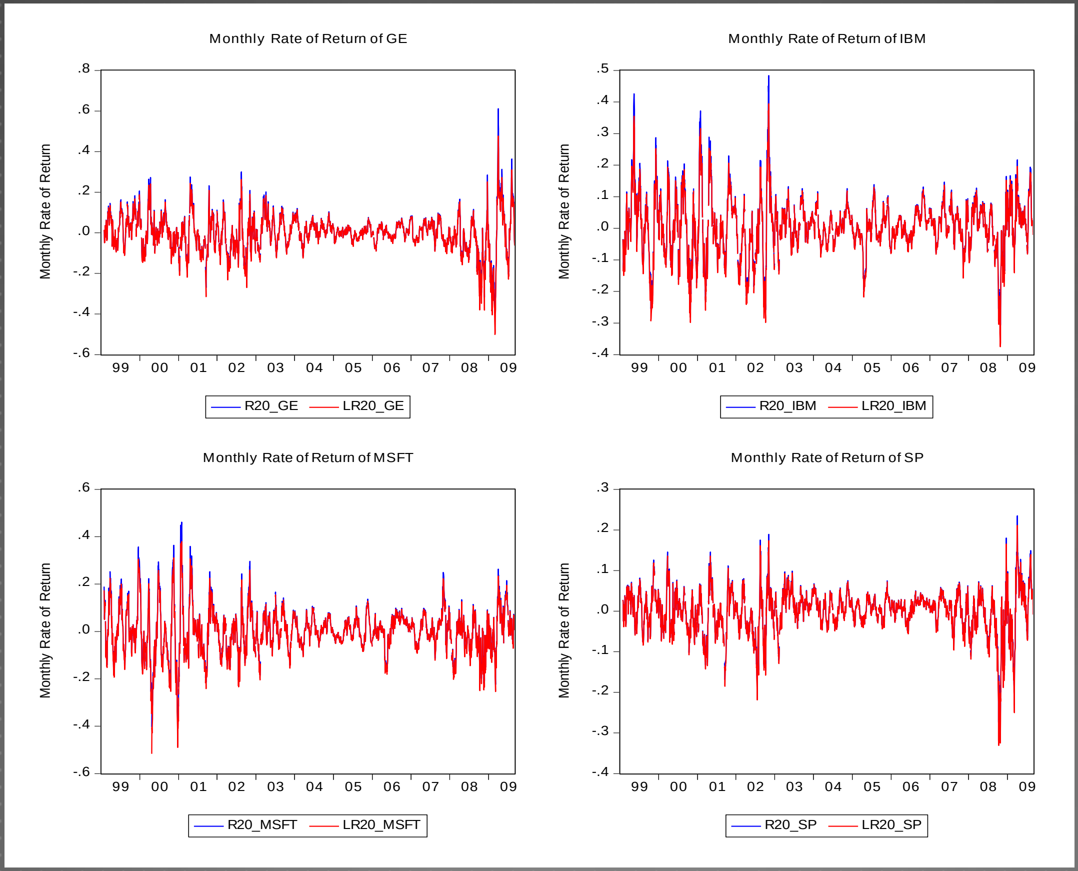
$$R_t = ln[1 + (S_t - S_{t-1})/S_{t-1}] = ln[1 + r_t]$$

if r_t *is small* $ln[1 + r_t] \approx r_t$

DAILY RETURNS



MONTHLY RETURNS



ASSET RETURNS

- One advantage of using log returns is given by time aggregation

  For example, the two-day return is equal to

$$R_{02} = ln[S_2 / S_0] = ln[(S_2 / S_1) \times (S_1 / S_0)] = ln[S_2 / S_1] + ln[S_1 / S_0]$$
$$= R_{01} + R_{12}$$

  More generally:

  $R_{t,t+k} = ln[S_{t+k} / S_t] = \Sigma R_{t+i}$

- Expected returns and variance for R_{02}:

  $E(R_{02}) = E(R_{01}) + E(R_{12})$

  $V(R_{02}) = V(R_{01}) + V(R_{12}) + 2Cov(R_{01}, R_{12})$

ASSET RETURNS

- Efficient Market Hypothesis:
  - Current prices convey all relevant information about the asset
  - Any change in the asset price is due to new news which are impossible to predict
  - This implies that changes in asset prices are unpredictable

- Random Walk

  $s_t = ln[S_t]$

  $$s_t = s_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim \phi(\mu,\sigma^2)$$

  $$\Delta s_t = R_t = \varepsilon_t$$

- If the distribution of ε_t is constant over time ε_t (and R_t) are independently and identically distributed (i.i.d.)

ASSET RETURNS

- Autocovariance function: $Cov(R_t, R_{t-i})$
- Autocorrelation function: $Corr(R_t, R_{t-i}) \equiv \rho(R_t, R_{t-i})$

- If we assume that returns are uncorrelated and i.i.d.

$$\rho(R_t, R_{t-i}) \approx 0 \quad for\ i = 1,2,3, \ldots T$$

then:

$$E(R_{01}) = E(R_{12}) = E(R_{23}) = \ldots \quad \rightarrow \quad E(R_{02}) = 2E(R_{01})$$

$$V(R_{01}) = V(R_{12}) = V(R_{23}) = \ldots \quad \rightarrow \quad V(R_{02}) = 2V(R_{01})$$

or more in general, over T days

$$E(R_T) = E(R_{01})T \equiv E(R_1)T \qquad (3.5)$$

$$V(R_T) = V(R_{01})T \equiv V(R_1)T \qquad (3.6)$$

$$SD(R_T) = SD(R_{01})\sqrt{T} \equiv SD(R_1)\sqrt{T} \qquad (3.7)$$

ASSET RETURNS

- When returns are uncorrelated (autocorrelation is zero for all lags), the volatility increases as the horizon increases, following the square root of time
- Autocorrelation function:

if $\rho(R_t, R_{t-i}) > 0$

movements in one direction one day are followed by movements in the same direction ➔ trend

if $\rho(R_t, R_{t-i}) < 0$

movements in one direction one day are followed by movements in the opposite direction ➔ mean reversion

STYLIZED FACTS OF ASSET RETURNS: MEAN AND STANDARD DEVIATION

- The standard deviation of returns dominates the mean of returns at short horizons such as daily
  - If we test the null hypothesis that the mean daily return is equal to zero, we fail to reject it!
  - Standard deviation of daily return is much higher than the mean

STYLIZED FACTS OF ASSET RETURNS: AUTOCORRELATION

- Daily returns have very little autocorrelation

$$\rho(R_t, R_{t-i}) \approx 0 \quad \text{for } i = 1,2,3, \ldots T$$

➔

Returns are impossible to predict from their own past

➔

Market efficiency!!!

STYLIZED FACTS OF ASSET RETURNS: SKEWNESS

- Stock market exhibits occasional very large drops but not equally large up-moves ➔ the distribution of asset returns is not symmetric

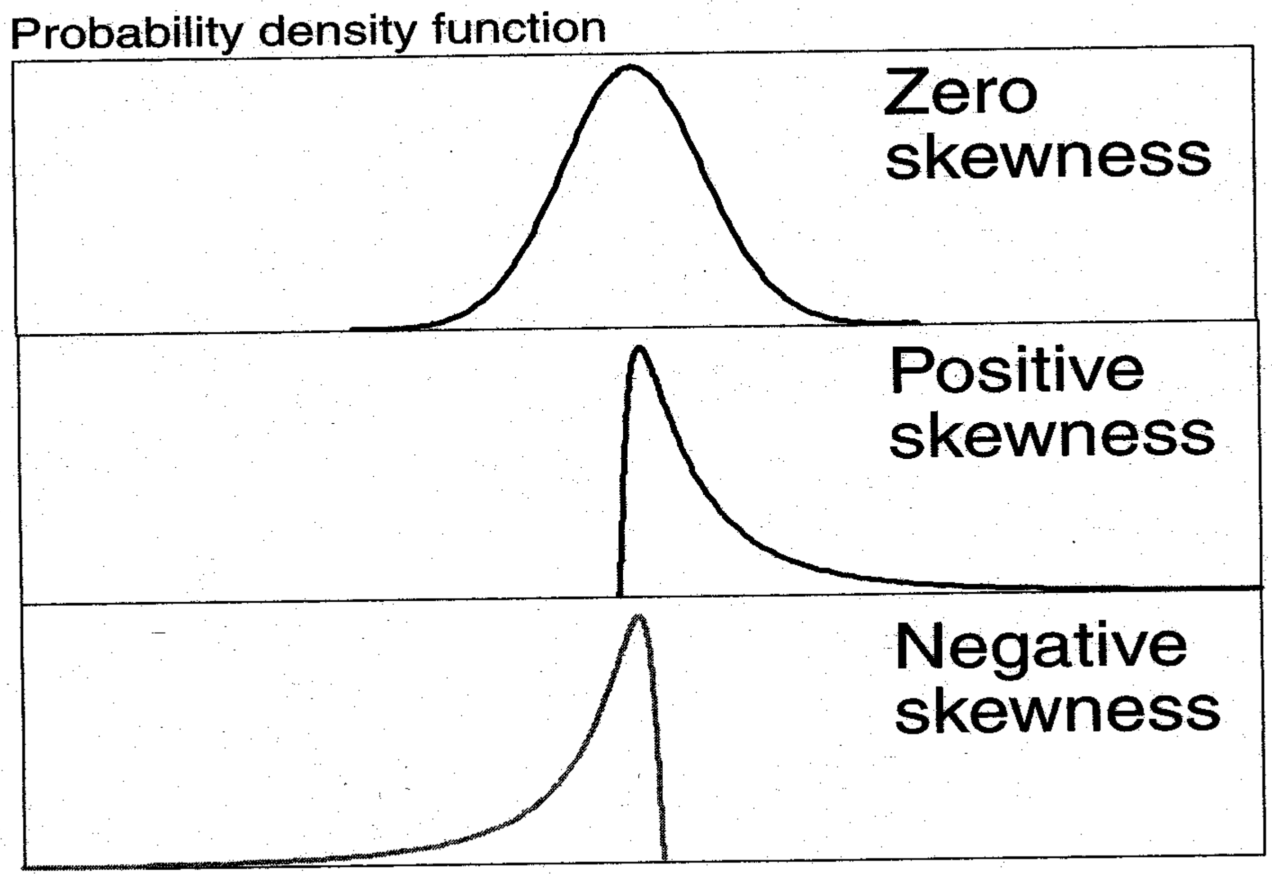
  Skewness: scaled third moment

$$\gamma = \left(\int_{-\infty}^{+\infty} \left[x - E(x) \right]^3 f(x) dx \right) / \sigma^3 \qquad (2.12)$$

- FX market tends to show less evidence of skewness

STYLIZED FACTS OF ASSET RETURNS: SKEWNESS

FIGURE 2-3 Effect of Skewness



STYLIZED FACTS OF ASSET RETURNS: KURTOSIS

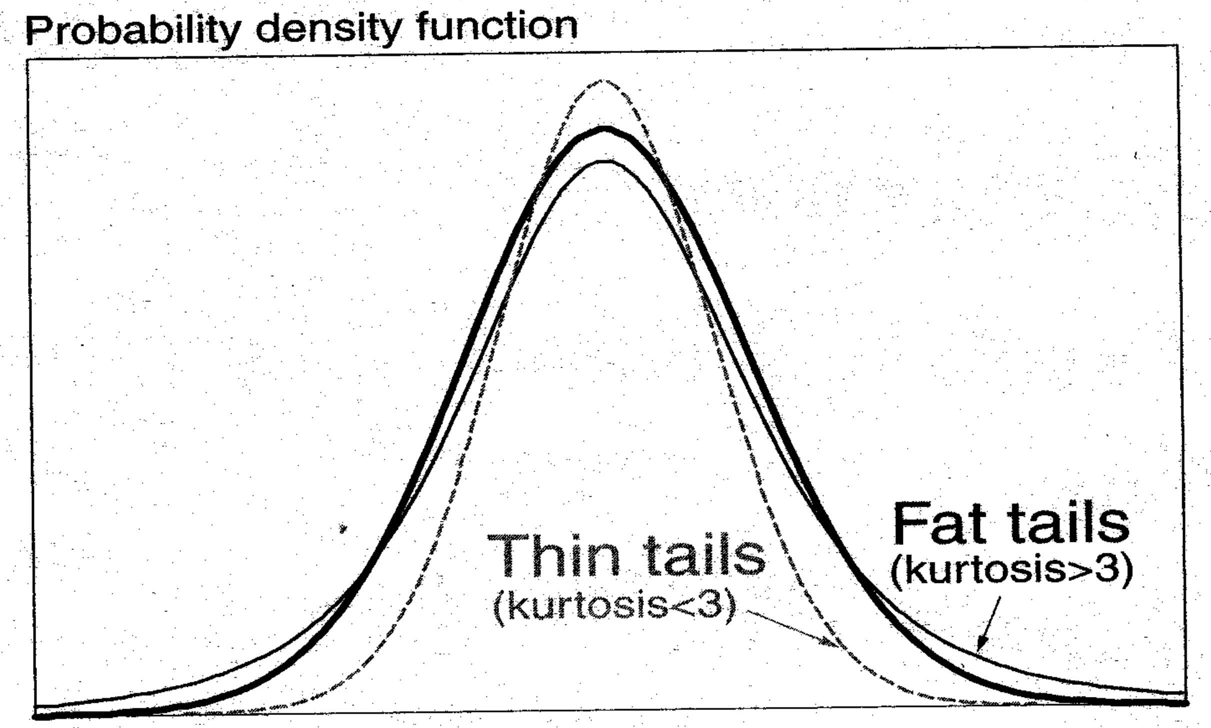
- The unconditional distribution of daily returns has fatter tails than the normal distribution ➔ higher probability of large losses

  Kurtosis: scaled fourth moment

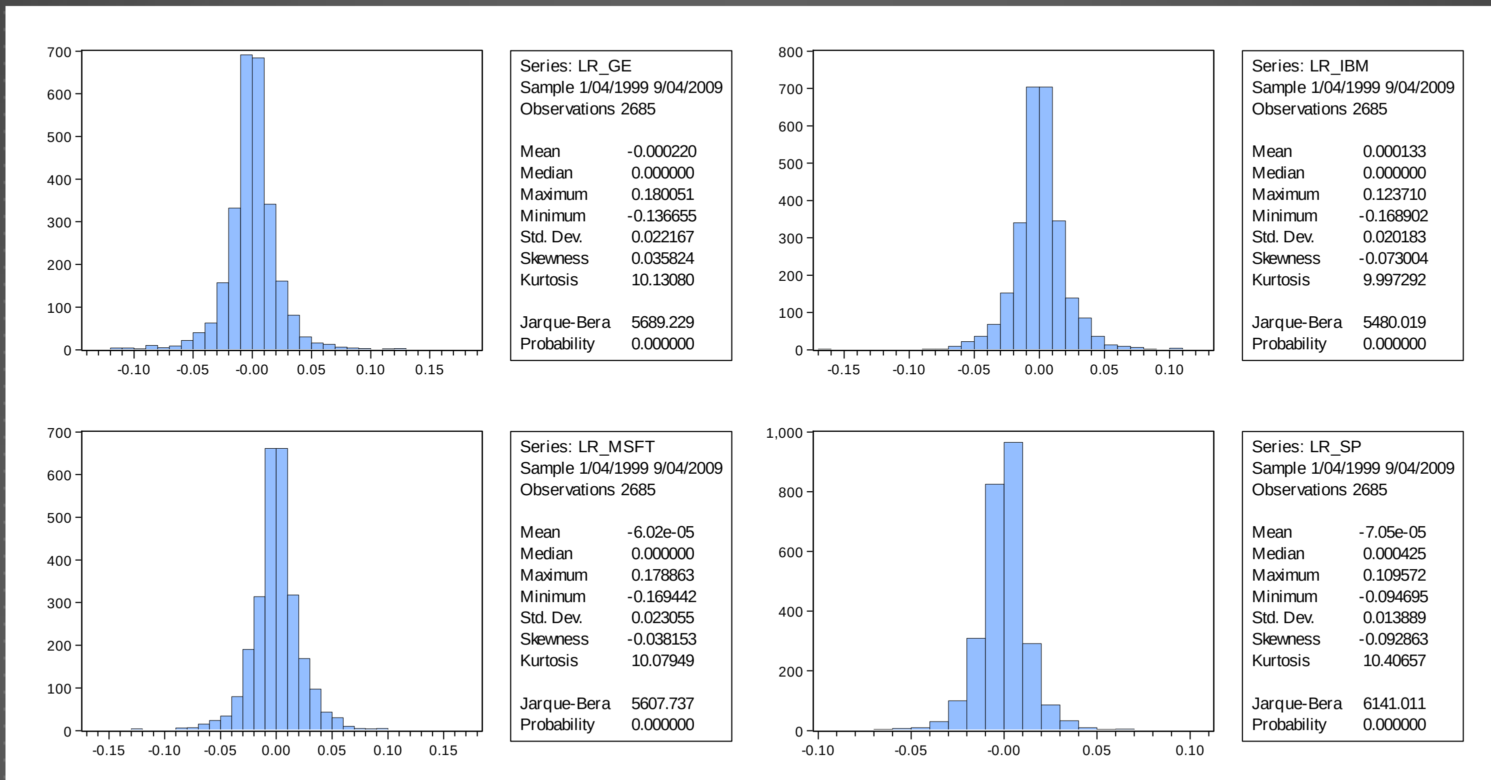
$$\delta = \left(\int_{-\infty}^{+\infty} \left[x - E(x) \right]^4 f(x) dx \right) / \sigma^4 \qquad (2.13)$$

STYLIZED FACTS OF ASSET RETURNS: KURTOSIS

FIGURE 2-4 Effect of Kurtosis



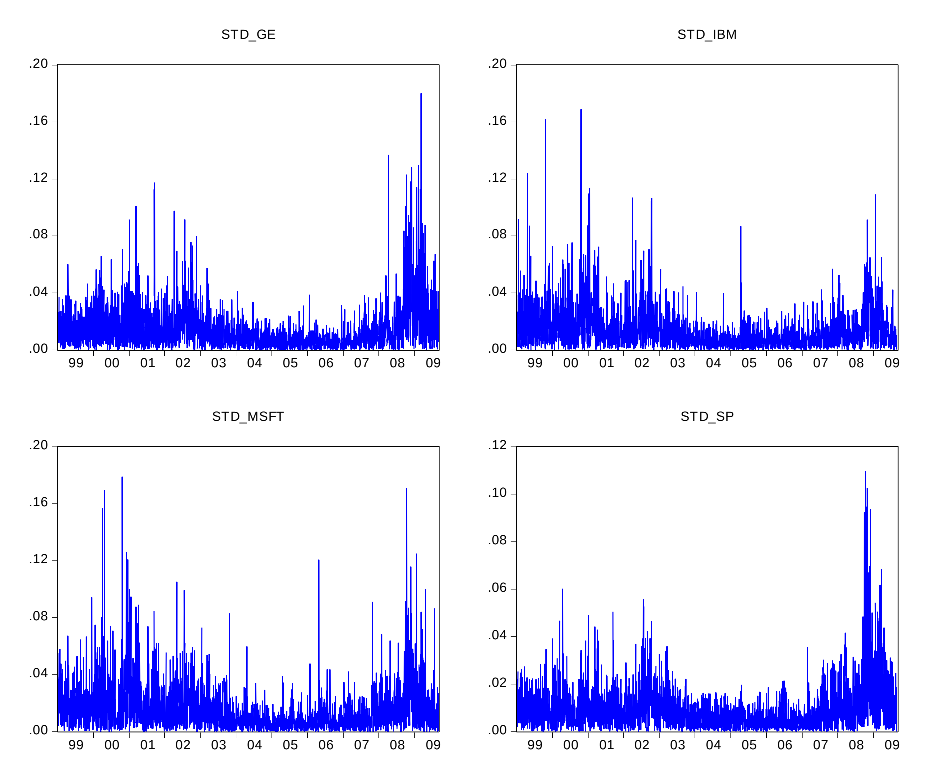
DESCRIPTIVE STATISTICS



STANDARD DEVIATION

- The standard deviation of returns dominates the mean of returns at short horizons. It is not possible to reject zero mean in short horizon.
- Standard deviations seem to be more volatile over time. It reaches the peak of 11% around the collapse of Lehman Brothers in September 2008.

STANDARD DEVIATIONS



STYLIZED FACTS OF ASSET RETURNS: SQUARED RETURNS

- Squared returns → variance

  $\sigma^2 = E(x^2) - [E(x)]^2$

  $\Delta s_t = R_t = \varepsilon_t \quad \varepsilon_t \sim \phi(0,\sigma^2) \qquad E(R_t) = 0$

  →

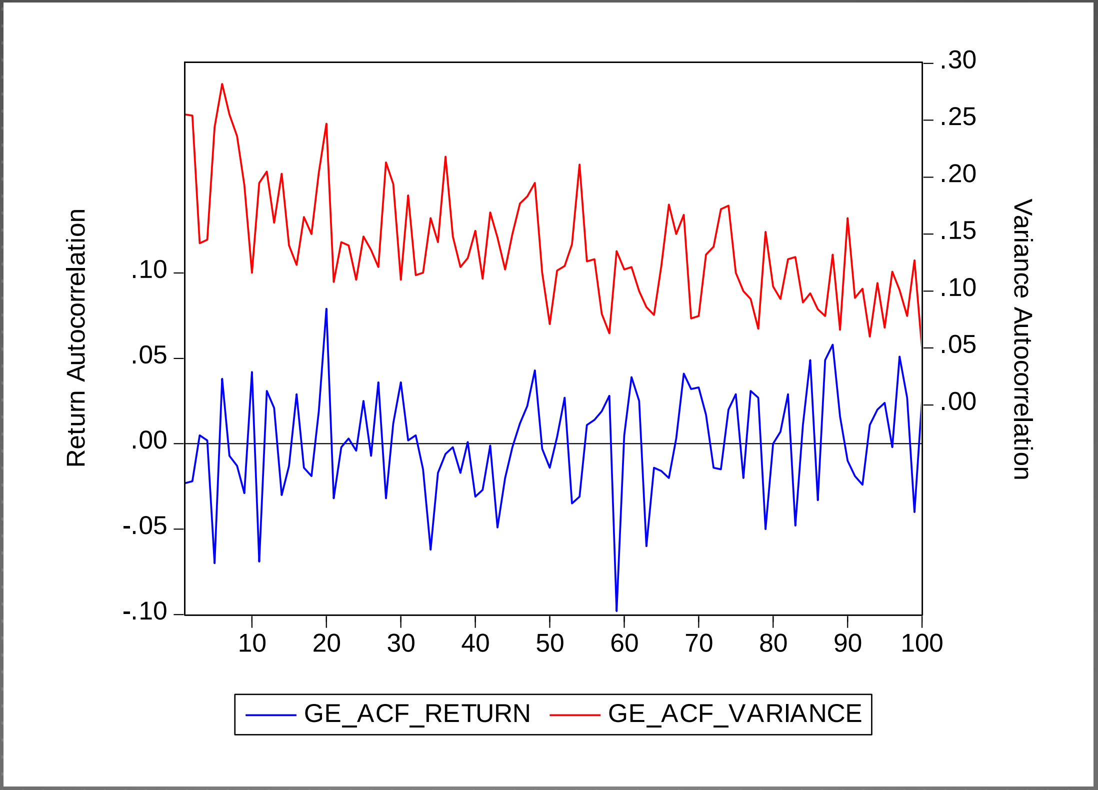
  $\sigma^2 = E(x^2)$

- Squared returns exhibit positive autocorrelation

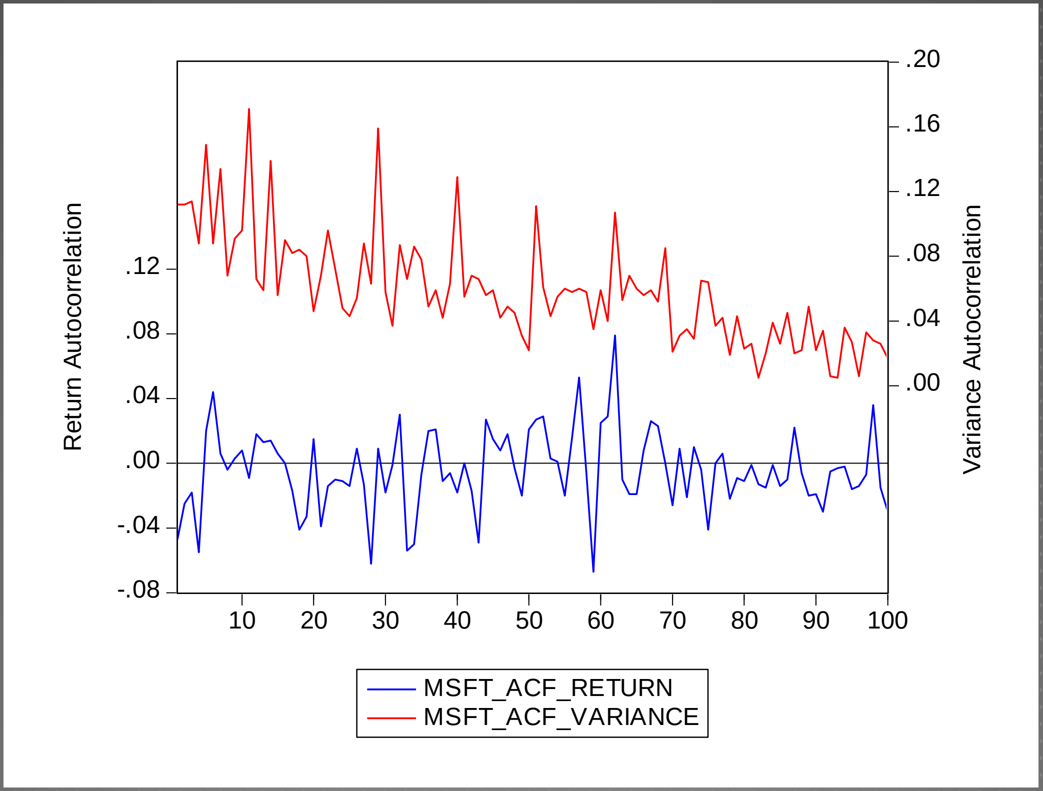
  $$\rho(R^2_t, R^2_{t-i}) > 0 \qquad for\ i = 1,2,3, \dots T$$

- The autocorrelations of squared returns tend to be positive for short lags and decay exponentially to zero as the number of lags increases.

AUTOCORRELATION FUNCTIONS: GE

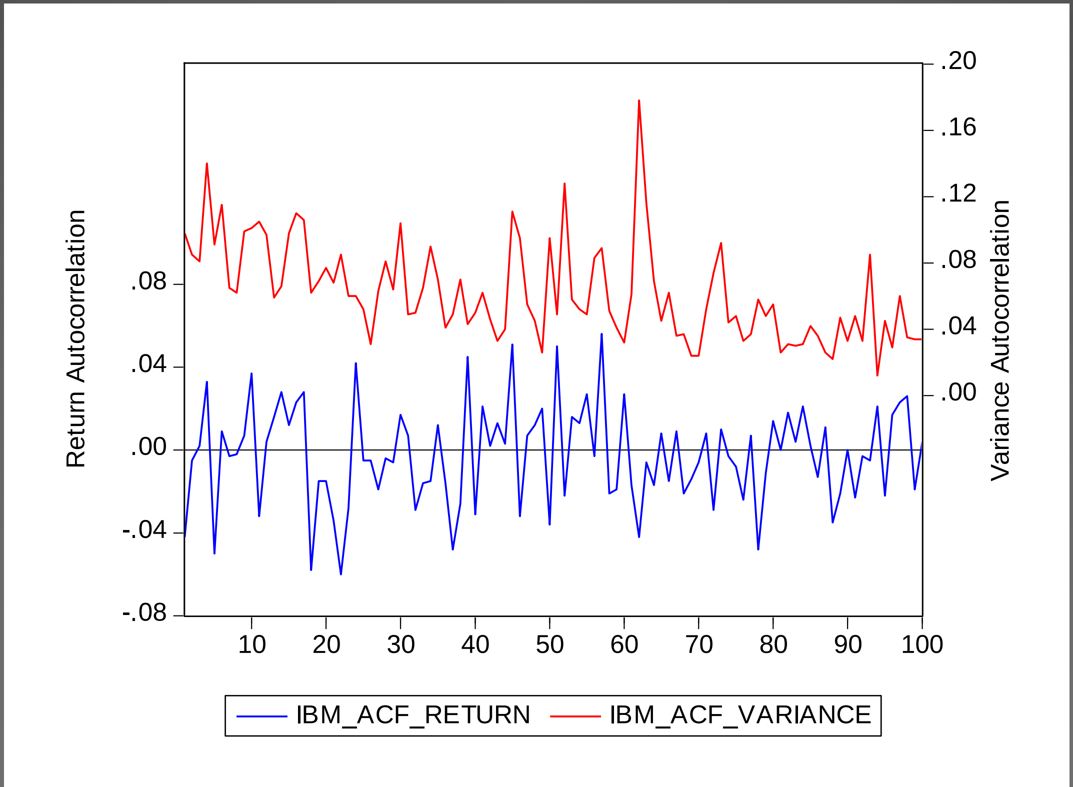


ACF: MSFT



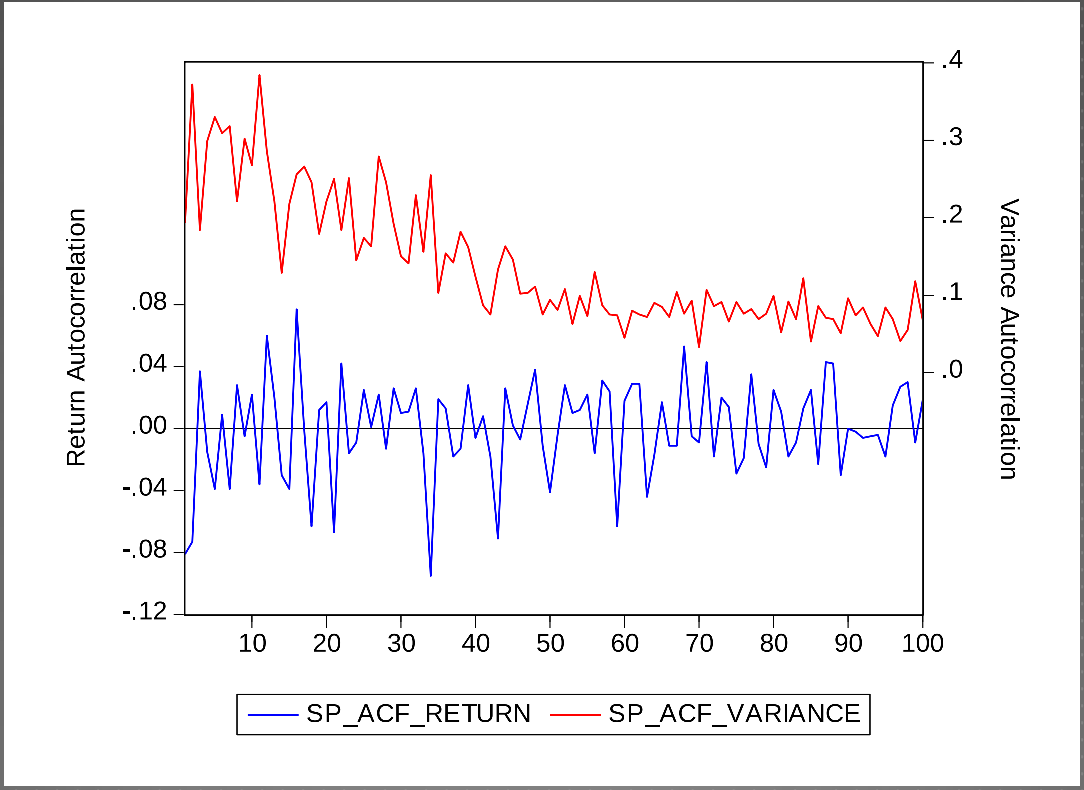
Return Autocorrelation
Variance Autocorrelation
.12
.08
.04
.00
-.04
-.08
.20
.16
.12
.08
.04
.00
10
20
30
40
50
60
70
80
90
100
MSFT_ACF_RETURN
MSFT_ACF_VARIANCE

ACF: IBM



Return Autocorrelation
Variance Autocorrelation
.08
.04
.00
-.04
-.08
.20
.16
.12
.08
.04
.00
10
20
30
40
50
60
70
80
90
100
IBM_ACF_RETURN
IBM_ACF_VARIANCE

ACF: S&P 500



Return Autocorrelation
Variance Autocorrelation
.08
.04
.00
-.04
-.08
-.12
.4
.3
.2
.1
.0
10
20
30
40
50
60
70
80
90
100
SP_ACF_RETURN
SP_ACF_VARIANCE

STYLIZED FACTS OF ASSET RETURNS: LEVERAGE EFFECT

- Equity and equity indices display negative correlation between variance and returns

➔

Leverage effect

A drop in the stock price will increase the leverage of the firm and therefore the risk (variance)

STYLIZED FACTS OF ASSET RETURNS: CORRELATION BETWEEN ASSETS

- Correlation between assets is not constant over time – i.e. it changes

  Empirical evidence shows that assets are more correlated during crashes!!!

  Covariance: $E(xy) = E[(x - E(x)) \times (y - E(y))]$

  $$\text{if } E(x) = 0 \quad \text{and} \quad E(y) = 0$$

  $$E(xy) = E(x \times y)$$

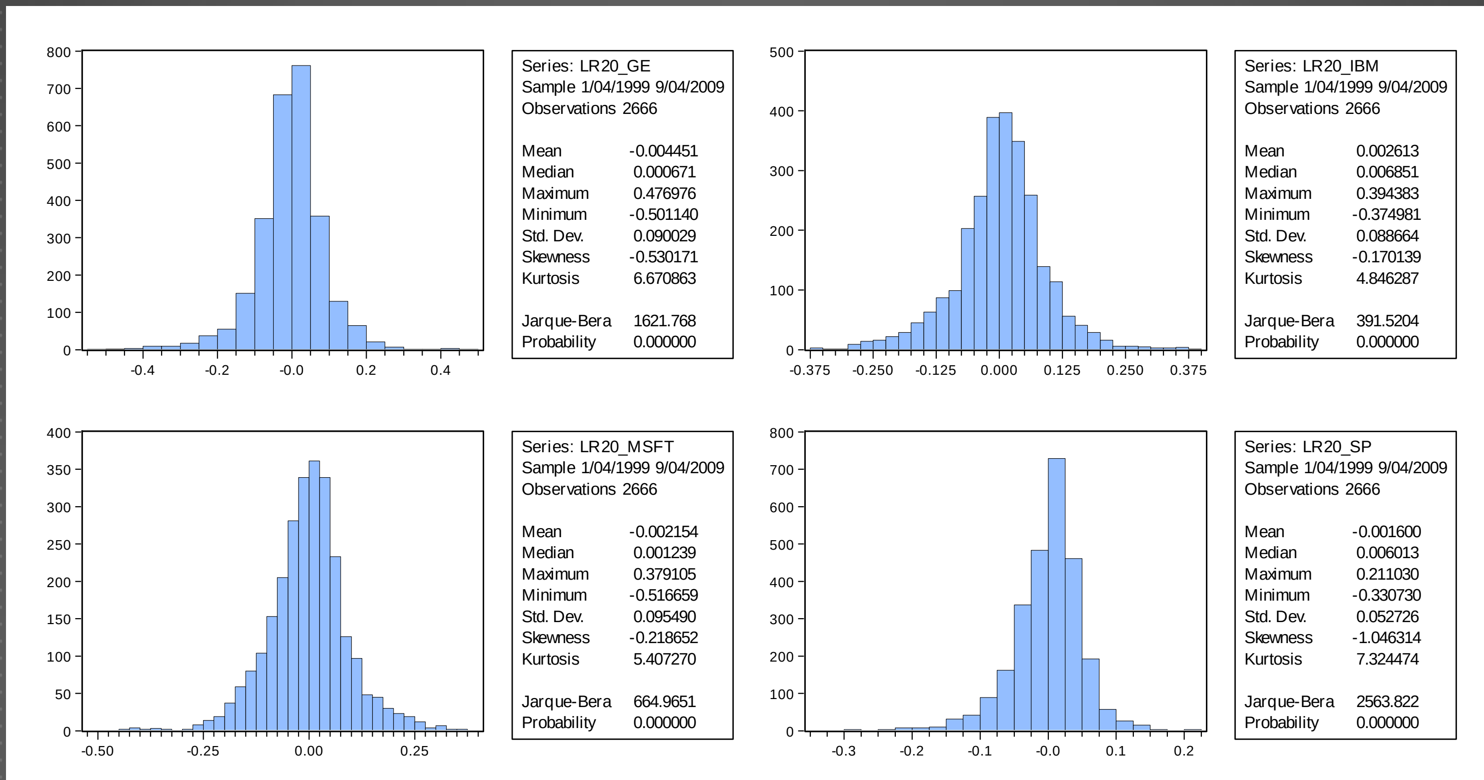
  $$Cov(R_{i,t}, R_{j,t}) = E(R_{i,t}, R_{j,t})$$

  Covariance between asset returns may be estimated by the product of the returns

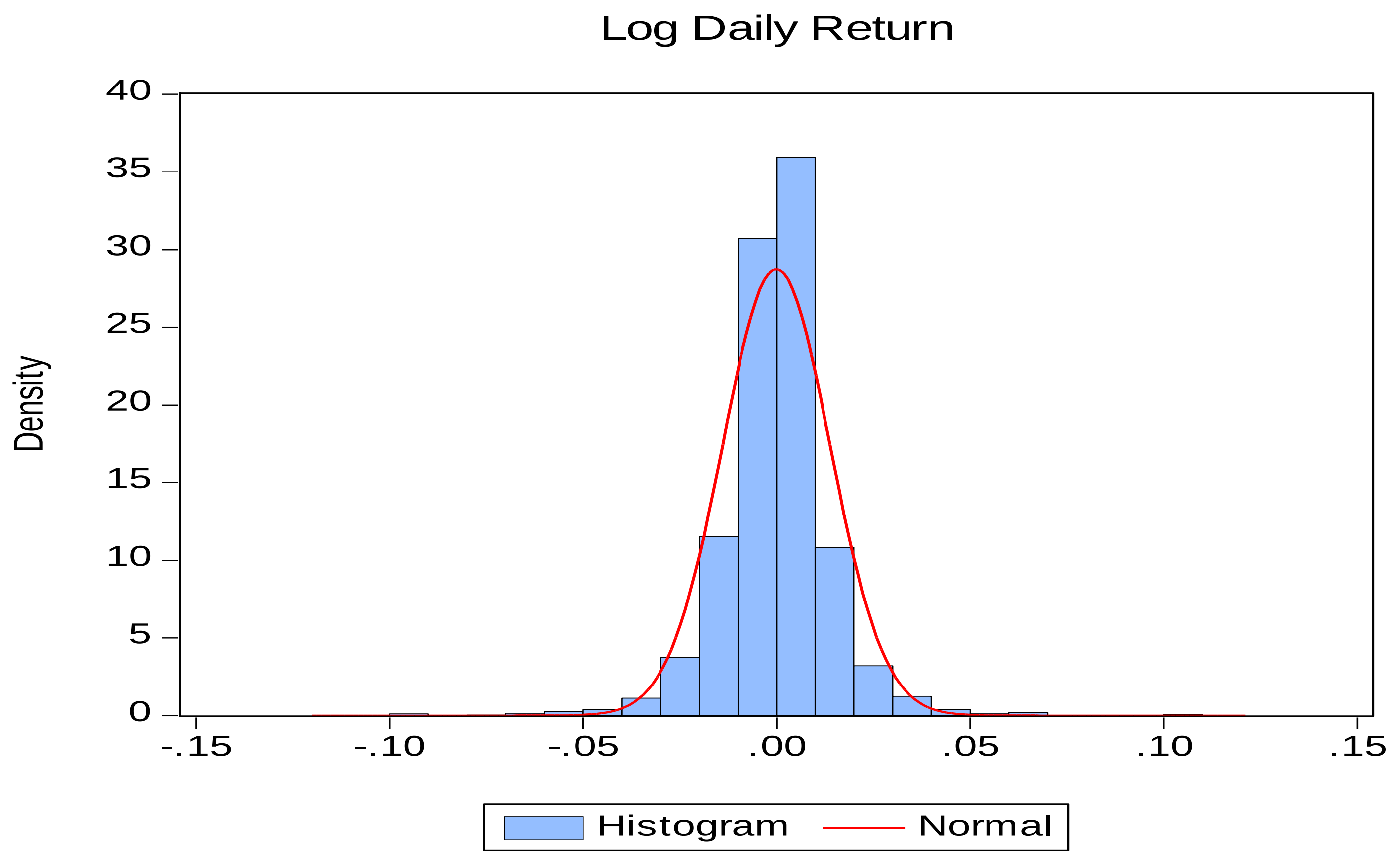
STYLIZED FACTS OF ASSET RETURNS: RETURN HORIZON

- As the return horizon increases, the unconditional return distribution changes and looks increasingly like a normal distribution

MONTHLY RETURNS



UNCONDITIONAL DISTRIBUTION DAILY RETURNS S&P500



MONTE CARLO
&
SIMULATION

RANDOM VARIABLES

MONTE CARLO SIMULATION

- Monte Carlo simulation involve creating artificial random variables, whose time path is not predictable.
- Any variable whose value changes over time in an unpredictable way is said to follow a *stochastic process*
- *A Markov process* is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future.

MARKOV PROCESSES

- If markets are efficient, financial prices should display random walk pattern
- More generally, in finance it is often assumed that asset prices follow a Markov process → the entire distribution of the future price relies on the current price only → past history is irrelevant
- Wiener process: describes a variable whose change is measured over an interval Δt, such that the average change is zero and its variance is proportional to Δt

$$\Delta z \sim N(0, \Delta t) \qquad (4.1)$$

$$or$$

$$\Delta z = \varepsilon\sqrt{\Delta t} \qquad \varepsilon \sim N(0,1)$$

MARKOV PROCESSES

- Generalized Wiener process: it contains a constant trend *a* and volatility *b*

$$\Delta x = a\Delta t + b\Delta z \qquad (4.2)$$

if a = 0 (no drift), the process is a Martingale ➔ expectation of the future value is the current value $E(x_T) = x_0$

- Ito process: trend (*a*) and volatility (*b*) depend on the current value of the underline variable and time

$$\Delta x = a(x,t)\Delta t + b(x,t)\Delta z \qquad (4.4)$$

This is a Markov process because the distribution depends only on the current value of x

GEOMETRIC BROWNIAN MOTION

- An example of Ito process is the GBM

$$\Delta S = \mu S \Delta t + \sigma S \Delta z \qquad (4.5)$$

- This process is geometric because trend and volatility are proportional to the current value of *S*.

- This process is particularly useful in modeling stock prices

μ represents the expected total rate of return (less dividend)

σ volatility

Example: A stock price process

Consider a stock that pays no dividends, has an expected return of 10% per annum, and volatility of 20% per annum. If the current price is \$100, what is the process for the change in the stock price over the next week? What if the current price is \$10?

The process for the stock price is

$$\Delta S = S(\mu \Delta t + \sigma \sqrt{\Delta t} \times \epsilon)$$

where ϵ is a random draw from a standard normal distribution. If the interval is one week, or $\Delta t = 1/52 = 0.01923$, the mean is $\mu \Delta t = 0.10 \times 0.01923 = 0.001923$ and $\sigma \sqrt{\Delta t} = 0.20 \times \sqrt{0.01923} = 0.027735$. The process is $\Delta S = \$100(0.001923 + 0.027735 \times \epsilon)$. With an initial stock price at \$100, this gives $\Delta S = 0.1923 + 2.7735\epsilon$. With an initial stock price at \$10, this gives $\Delta S = 0.01923 + 0.27735\epsilon$. The trend and volatility are scaled down by a factor of 10.

Example: A stock price process (continued)

Assume the price in one week is given by $S = \$100 \exp(R)$, where R has annual expected value of 10% and volatility of 20%. Construct a 95% confidence interval for S.

The standard normal deviates that corresponds to a 95% confidence interval are $\alpha_{\text{MIN}} = -1.96$ and $\alpha_{\text{MAX}} = 1.96$. In other words, we have 2.5% in each tail. The 95% confidence band for R is then $R_{\text{MIN}} = \mu\Delta t - 1.96\sigma\sqrt{\Delta t} = 0.001923 - 1.96 \times 0.027735 = -0.0524$ and $R_{\text{MAX}} = \mu\Delta t + 1.96\sigma\sqrt{\Delta t} = 0.001923 + 1.96 \times 0.027735 = 0.0563$. This gives $S_{\text{MIN}} = \$100 \exp(-0.0524) = \$94.89$, and $S_{\text{MAX}} = \$100 \exp(0.0563) = \$105.79$.

GEOMETRIC BROWNIAN MOTION

- This is an important model: used in Black-Scholes
- Volatility is proportional to *S* ➔the stock price will stay positive,

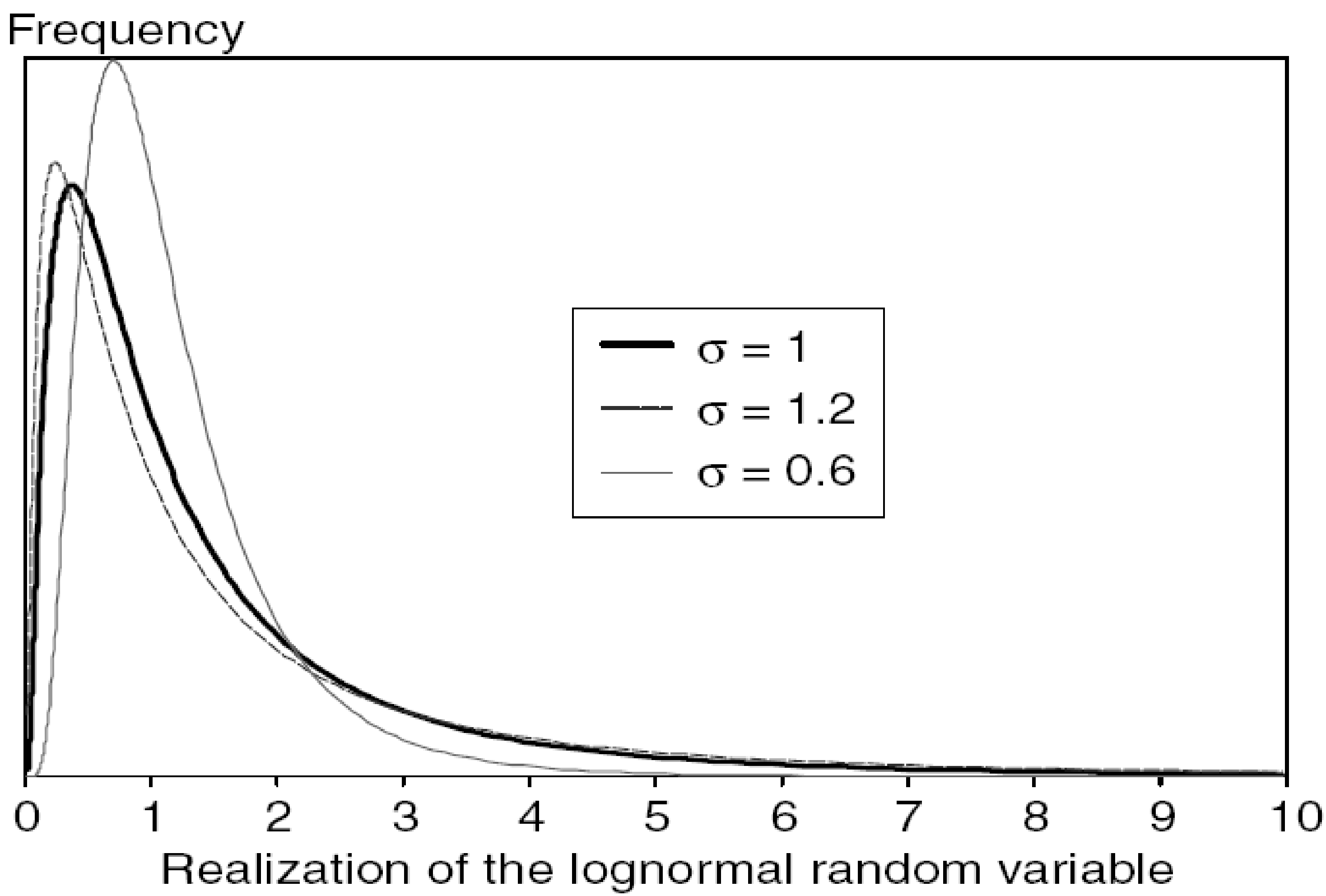
  If the stock price falls ➔ its variance decreases ➔ it is very unlikely that there will be a down large movement that makes the stock negative
- At the limit, when $\Delta t \to 0$ (very small intervals/increments)

$$dS/S = d\ln(S)$$

- Given that $dS/S = d\ln(S)$ follows a Normal distribution ➔ S follows a log-normal distribution

  In fact if $Y = \ln(X)$ is normally distributed, then *X* follows a log-normal distribution

FIGURE 2-8 Lognormal Density Function



GEOMETRIC BROWNIAN MOTION

- The GBM implies that over an interval of time, T – t = τ the *ln* of the final price is given by

$$\ln(S_T) = \ln(S_t) + (\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}\,\epsilon \qquad (4.6)$$

- Should we choose a Normal distribution or a log-Normal for our price process? It depends on the time horizon
  - Short time periods (such as a day) it does not matter which of the two distributions we choose
  - Long horizons (such as years) it is better to adopt the log-normal ➔ prices will not be negative

SIMULATION

Wiener Process

$$\Delta S = \varepsilon\sqrt{\Delta t} \qquad \varepsilon \sim N(0,1) \qquad S_{t+1} = S_t + \Delta S = S_t + \varepsilon_{t+1}\sqrt{\Delta t}$$

$$\Delta t = \frac{1}{252} \quad \textit{daily} \implies 252 \textit{ steps (simulating one year)}$$

Generalized Wiener Process

$$\Delta S = a\Delta t + b\varepsilon\sqrt{\Delta t} \qquad S_{t+1} = S_t + \Delta S = S_t + a\Delta t + b\varepsilon_{t+1}\sqrt{\Delta t}$$

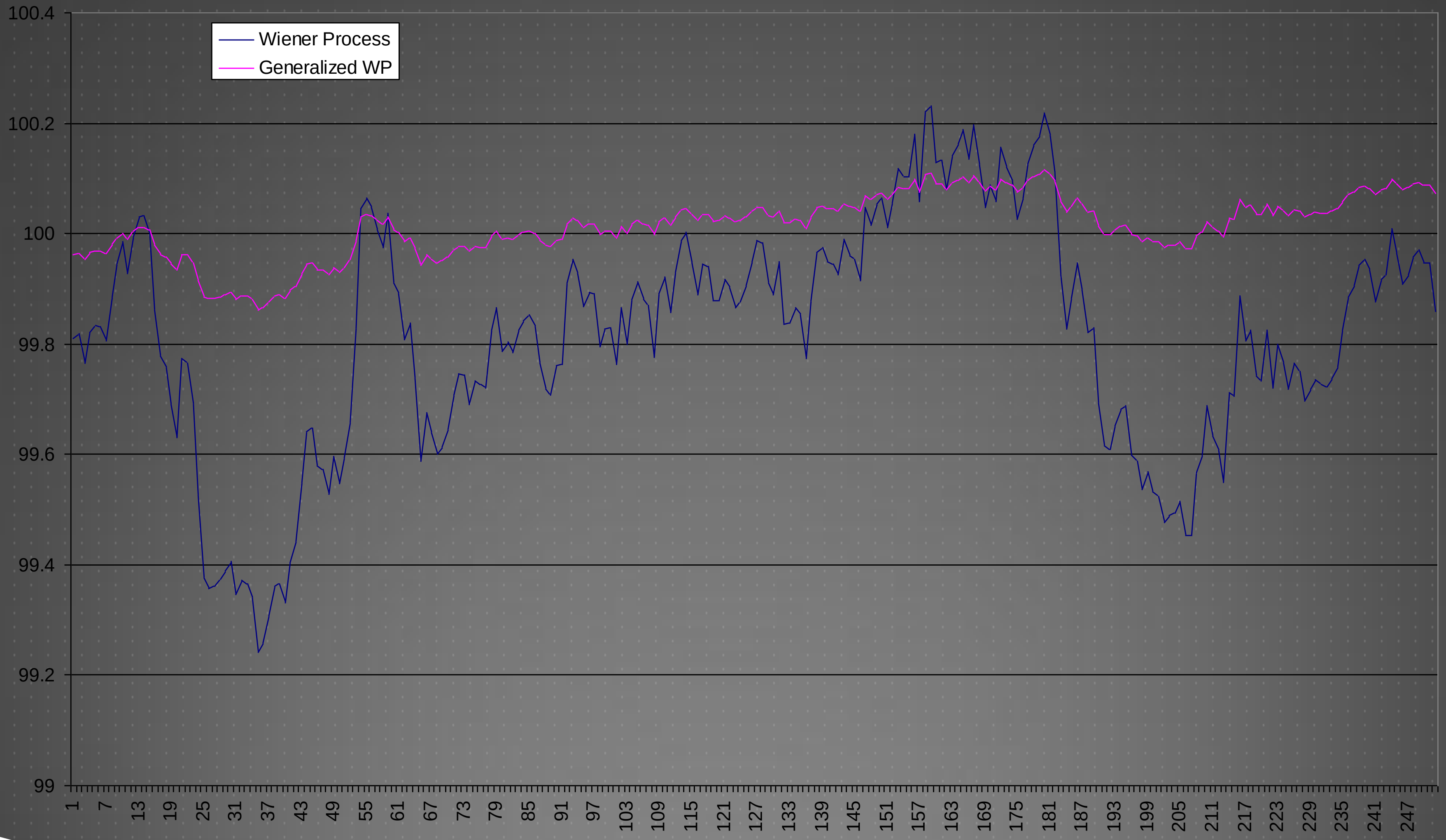
$$a = 10\% \text{ per annum},\ b = 20\% \text{ per annum}$$

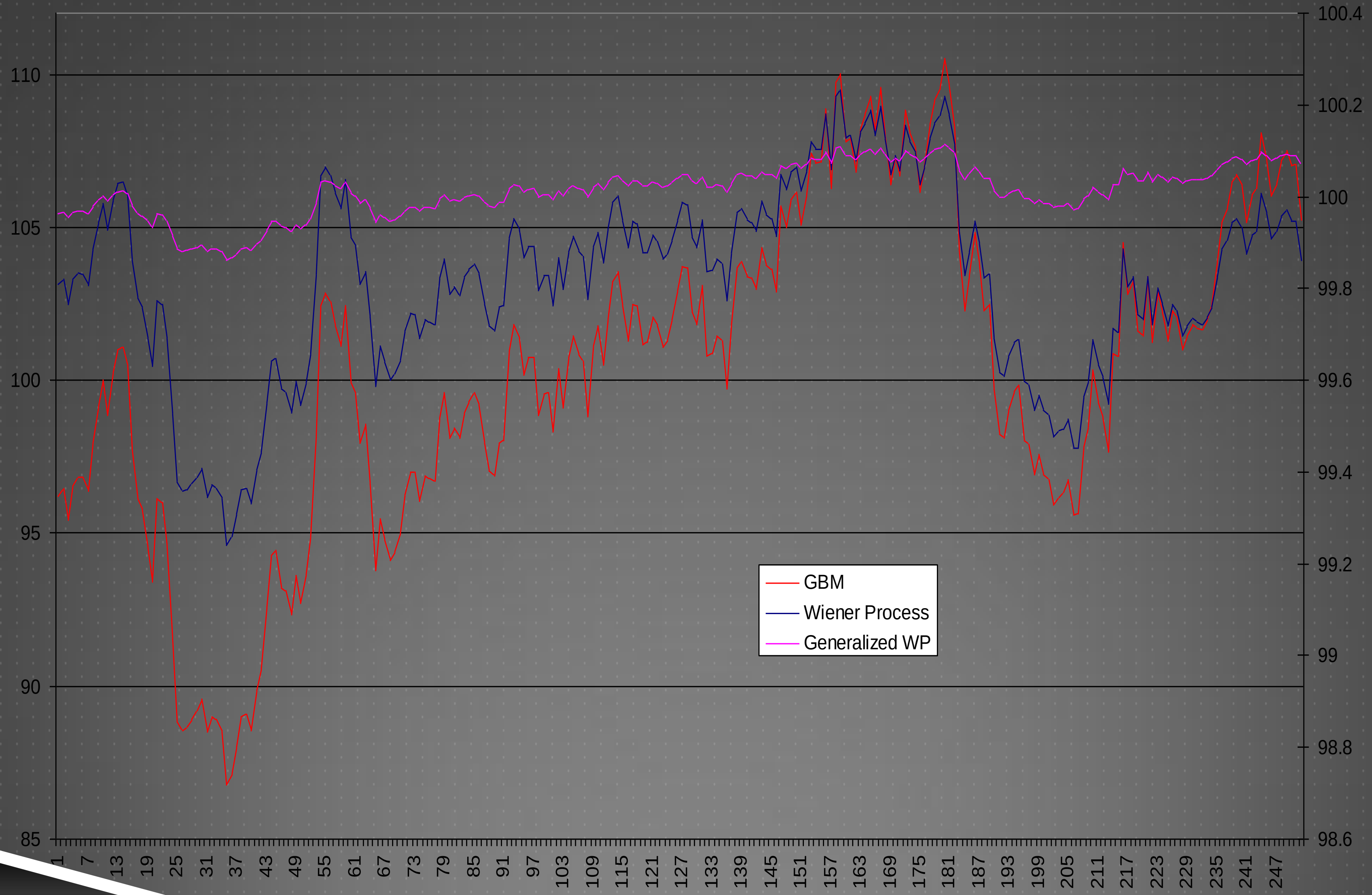
Geometric Brownian Motion

$$\Delta S = \mu S\Delta t + \sigma S\varepsilon\sqrt{\Delta t} \qquad \frac{\Delta S}{S} = \mu\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

$$\frac{\Delta S}{S} \sim N\left(\mu\Delta t, \sigma^2\Delta t\right) \qquad (4.7)$$

$$S_{t+1} = S_t + \Delta S = S_t + S_t\left(\mu\Delta t + \sigma\varepsilon_{t+1}\sqrt{\Delta t}\right)$$


Wiener Process
Generalized WP
100.4
100.2
100
99.8
99.6
99.4
99.2
99
1 7 13 19 25 31 37 43 49 55 61 67 73 79 85 91 97 103 109 115 121 127 133 139 145 151 157 163 169 175 181 187 193 199 205 211 217 223 229 235 241 247



110
105
100
95
90
85
100.4
100.2
100
99.8
99.6
99.4
99.2
99
98.8
98.6
GBM
Wiener Process
Generalized WP

Drift (Annual)		10%
Volatility (Annual)		40%
Drift (daily)		0.04%
Volatilty (daily)		2.52%
Drift (mean)		0.01%
Initial Stock	$	100.00

$$ln(P_t/P_{t-1}) = \mu + \sigma z_t$$

	1	2	3	4	5	6	7	8	9	10
N(0,1)	-0.944415595	-0.16806	-0.59763	0.976508	-0.35888	-0.75882	-0.55818	-0.2774	0.309081	-1.32457
Log Return	-2.37%	-0.42%	-1.50%	2.47%	-0.90%	-1.90%	-1.40%	-0.69%	0.79%	-3.33%
Price	97.65613823	97.25118	95.80526	98.19965	97.32337	95.48777	94.16162	93.51316	94.25177	91.16516

$$ln(\frac{P_t}{P_{t-1}}) \rightarrow N[(\alpha - \frac{\sigma^2}{2})T, \sigma\sqrt{T}]$$

SIMULATION

- Remember:

  We can replicate a Monte Carlo experiment many times. Each time we will generate a new path

  Initial values may affect the results

  Seed: it allows to generate the same random numbers

TERM STRUCTURE OF INTEREST RATE

- Bond prices display mean reversion to the face value

  We can first model the yield (interest rate) and then the bond price ➔ mean reversion in the yield

  GBM do not display mean reversion

  One factor models

$$\Delta r_t = \kappa(\theta - r_t)\Delta t + \sigma r_t^{\gamma}\,\Delta z_t \qquad (4.8)$$

  $\theta \geq 0$, is the long-run value of the interest rate *r*

  $\kappa \in [0,1]$, is the speed of mean reversion

  $\gamma = 0$ ➔ Vasicek Model (JFE - 1977)

  $\gamma = 0.5$ ➔ Cox, Ingersoll and Ross (CIR), (Econometrica 1985)

TERM STRUCTURE OF INTEREST RATE

- In C-I-R's model the stochastic term is proportional to $r^{1/2}$ indicating that, as the short term interest rate increases, its standard deviation increases
- In C-I-R's model interest rates are always positive. A negative value is implausible for nominal interest rates but not for real rates
- In Vasicek the short term interest rate can be negative: implausible for nominal but not for real rates
- Vasicek and C-I-R both accommodate rising, falling and single humped structure
- In both Vasicek and C-I-R the short term interest rate converges towards a long term interest rate which is constant and independent from r
- However, in practice, long rate is quite variable ➔ two-factor models

BINOMIAL TREE

- Two steps: up or down
- Binomial model ➔Discrete version of a GBM

Divide the horizon *T* into *n* intervals: $\Delta t = T/n$

At each node the price can go up with probability *p* or down with probability *1 – p*

Option evaluation with binomial tree

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = (1/u), \quad p = \frac{e^{\mu\Delta t} - d}{u - d} \tag{4.11}$$

$$E\left[\frac{S_1}{S_0}\right] = pu + (1-p)d = \frac{e^{\mu\Delta t} - d}{u - d}u + \frac{u - e^{\mu\Delta t}}{u - d}d$$

$$= \frac{e^{\mu\Delta t}(u-d) - du + ud}{u - d} = e^{\mu\Delta t}$$

BINOMIAL TREE

TABLE 4-2 Binomial Tree

		Step				
0		1		2		3
						u^3S
					↗	
				u^2S		
			↗		↘	
		uS				u^2dS
	↗		↘		↗	
S				udS		
	↘		↗		↘	
		dS				d^2uS
			↘		↗	
				d^2S		
					↘	
						d^3S

IMPLEMENTING SIMULATIONS FOR VALUE AT RISK

1. Choose stochastic Process → distribution and parameters
2. Generate pseudo-random variables at the target horizon, S_T
3. Calculate the value at the target horizon, $F_T(S_T)$
4. Repeat steps 2 and 3, *K* times → $F^1_T, F^2_T, F^3_T, \ldots, F^K_T$
5. Distribution of F_T → VaR

IMPLEMENTING SIMULATIONS

- Simulations for option pricing: Risk-neutral approach ➔ the price grows at the risk-free rate ➔ model based simulation

- Simulations for VaR: physical distribution – i.e. the actual distribution

- Sampling variability: even with *K* large, the empirical distribution of S_T will only be an approximation of the true distribution

  Monte Carlo: Independent draws ➔ standard error is inversely related to $\sqrt{K}$

MULTIPLE SOURCES OF RISK

- N number of risk factors, $j = 1, 2, 3, \ldots N$

$$\Delta S_{j,t} = S_{j,t-1}\mu_j \Delta t + S_{j,t-1}\sigma_j \epsilon_{j,t}\sqrt{\Delta t} \qquad (4.14)$$

- Risk factors are usually correlated

$$\begin{aligned} \epsilon_1 &= \eta_1 \\ \epsilon_2 &= \rho\eta_1 + (1-\rho^2)^{1/2}\eta_2 \end{aligned} \qquad (4.15)$$

$$V(\epsilon) = \begin{bmatrix} \sigma^2(\epsilon_1) & \mathrm{Cov}(\epsilon_1, \epsilon_2) \\ \mathrm{Cov}(\epsilon_1, \epsilon_2) & \sigma^2(\epsilon_2) \end{bmatrix} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = R$$

MULTIPLE SOURCES OF RISK

- If we have *N* risk factors, we have $N(N-1)/2$ covariances
- A portfolio with 100 assets will have to consider 4,950 covariances!
- We only consider the main risk factors