

# HEDGING NONLINEAR RISK

A decorative graphic at the bottom of the slide consists of a solid green trapezoidal shape pointing downwards, which is partially overlapped by a yellow trapezoidal shape pointing upwards. Both shapes have a white border.

# LINEAR AND NONLINEAR HEDGING

- ▶ Linear hedging → forwards and futures → values are linearly related to the underlying risk factors

Because linear combinations of normal random variables are also normally distributed, linear hedging maintains normal distributions

- ▶ Nonlinear hedging → Options

Since options can be replicated by dynamic trading of the underlying instruments, this also provides insights into the risks of active trading strategies

# OPTIONS: NOTATION

$S_t$  = current spot price of the asset in dollars

$F_t$  = current forward price of the asset

$K$  = exercise price of the option (also called strike price)

$f_t$  = current value of the derivative instrument

$r_t$  = domestic risk-free rate

$r_t^*$  = foreign risk free rate (also written as  $y$ : income produced by the asset)

$\sigma_t$  = annual volatility of the rate of change (returns) in  $S$

$\tau = T - t$  = time to maturity

For most options:

$$f_t = f(S_t, r_t, r_t^*, \sigma_t, K, \tau)$$

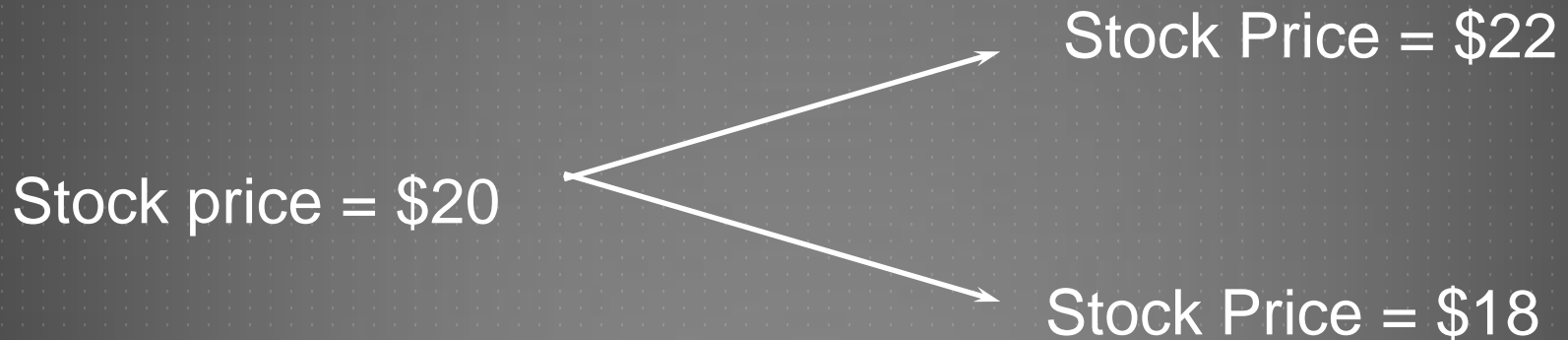
# TAYLOR EXPANSION

$$df = \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial r^*}dr^* + \frac{\partial f}{\partial \sigma}d\sigma + \frac{\partial f}{\partial \tau}d\tau + \dots \quad (13.3)$$

- ▶ **Option pricing** is about finding  $f$
- ▶ **Option hedging** uses the partial derivatives
- ▶ **Risk management** is about combining those with the movements in the risk factors
  
- ▶ **Remember:** the Taylor approximation works only for small movements in the underlying risk factor

# A SIMPLE BINOMIAL MODEL

- ▶ A stock price is currently \$20
- ▶ In three months it will be either \$22 or \$18



# A CALL OPTION

A 3-month call option on the stock has a strike price of 21.

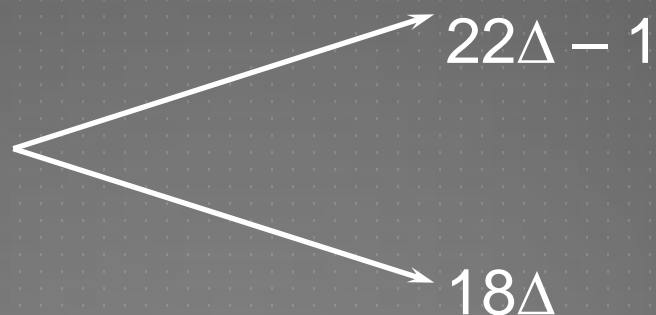
Stock price = \$20  
Option Price = ?

Stock Price = \$22  
Option Price = \$1

Stock Price = \$18  
Option Price = \$0

# SETTING UP A RISKLESS PORTFOLIO

- ▶ Consider the Portfolio: long position in  $\Delta$  shares of the stock and short 1 call option



- ▶ Portfolio is riskless when  $22\Delta - 1 = 18\Delta$  or  $\Delta = 0.25$

# VALUING THE PORTFOLIO (RISK-FREE RATE IS 12%)

- ▶ The riskless portfolio is:
  - long 0.25 shares
  - short 1 call option
- ▶ The value of the portfolio in 3 months is  
 $22 * 0.25 - 1 = 4.50 = 18 * 0.25$
- ▶ The value of the portfolio today is  
 $4.5e^{-0.12 * 0.25} = 4.3670$

# VALUING THE OPTION

- ▶ The portfolio that is  
long 0.25 shares short 1 option is worth 4.367 today
- ▶ The value of the portfolio today will be  
 $20 * 0.25 - f = 4.367$

where  $f$  represent option price today.

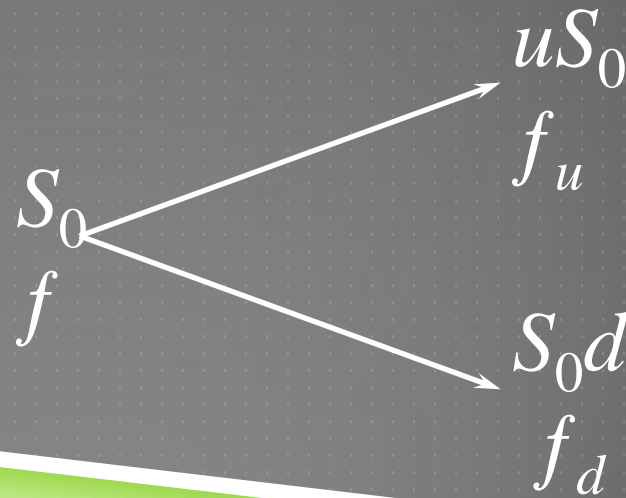
- ▶ The value of the option is therefore

$$f = 5.000 - 4.367$$

$$f = 0.633$$

# GENERALIZATION

- ▶ A derivative lasts for time  $T$  and is dependent on a stock



# GENERALIZATION (CONTINUED)

- ▶ Consider the portfolio that is long  $\Delta$  shares and short 1 derivative

$$S_0 - f \begin{cases} \rightarrow S_0 u \Delta - f_u \\ \rightarrow S_0 d \Delta - f_d \end{cases}$$

- ▶ The portfolio is riskless when  $S_0 u \Delta - f_u = S_0 d \Delta - f_d$  or

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

# GENERALIZATION (CONTINUED)

- ▶ Value of the portfolio at time  $T$  is  $S_0 u \Delta - f_u$
- ▶ Value of the portfolio today is  $(S_0 u \Delta - f_u) e^{-rT}$
- ▶ Another expression for the portfolio value today is  $S_0 \Delta - f$
- ▶ Hence
$$f = S_0 \Delta - (S_0 u \Delta - f_u) e^{-rT}$$

# GENERALIZATION (CONTINUED)

- ▶ Substituting for  $\Delta$  we obtain

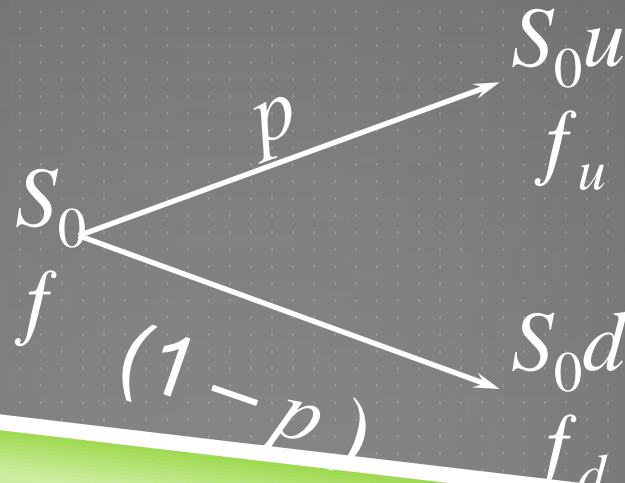
$$f = [p f_u + (1 - p) f_d] e^{-rT}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

# RISK-NEUTRAL VALUATION

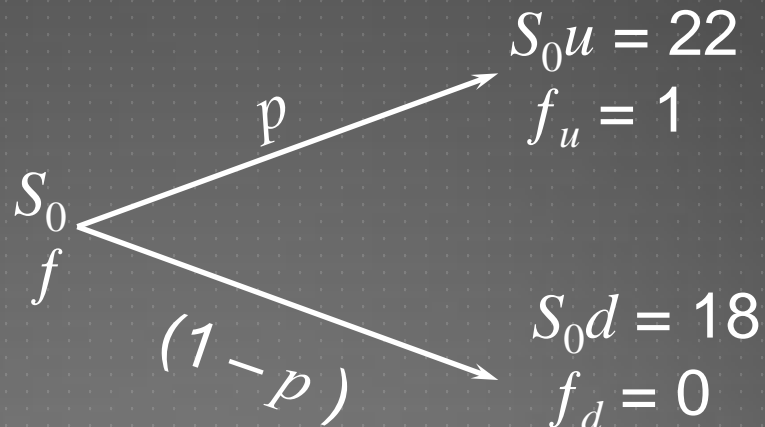
- ▶  $f = [p f_u + (1 - p) f_d] e^{-rT}$
- ▶ The variables  $p$  and  $(1 - p)$  can be interpreted as the risk-neutral probabilities of up and down movements
- ▶ The value of a derivative is its expected payoff in a risk-neutral world discounted at the risk-free rate



# IRRELEVANCE OF STOCK'S EXPECTED RETURN

- ▶ When we are valuing an option in terms of the underlying stock the expected return on the stock is irrelevant
- ▶ This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant

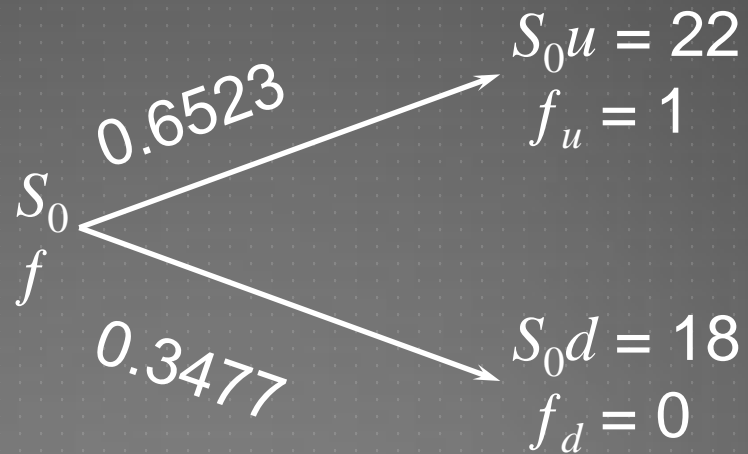
# ORIGINAL EXAMPLE REVISITED



- ▶ Since  $p$  is a risk-neutral probability
- ▶  $20e^{0.12 \cdot 0.25} = 22p + 18(1-p)$ ;  $p = 0.6523$
- ▶ Alternatively, we can use the formula

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

# VALUING THE OPTION

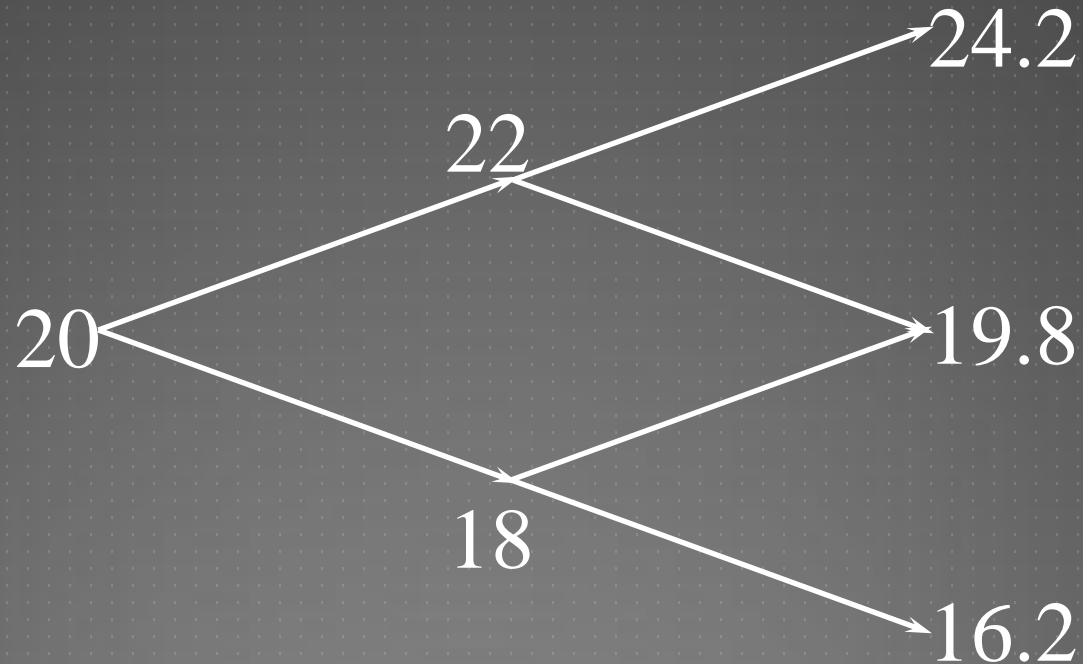


The value of the option is

$$e^{-0.12 \cdot 0.25} [0.6523 \cdot 1 + 0.3477 \cdot 0] = 0.633$$

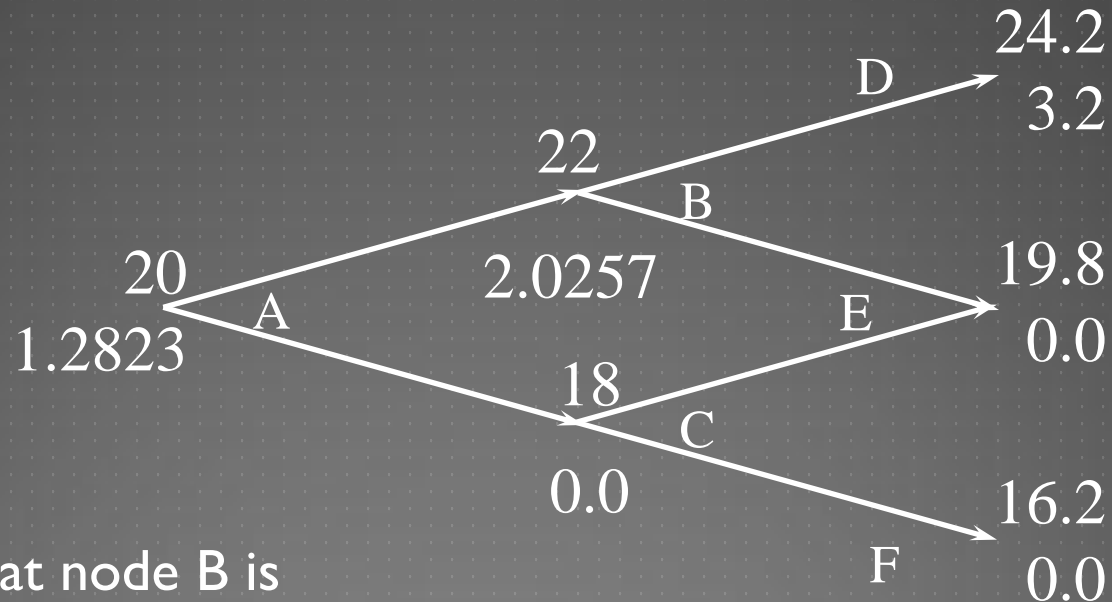
# A TWO-STEP EXAMPLE

FIGURE 11.3, PAGE 242



- ▶ Each time step is 3 months
- ▶  $K=21$ ,  $r=12\%$

# VALUING A CALL OPTION



- ▶ Value at node B is

$$e^{-0.12 \times 0.25} (0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$$

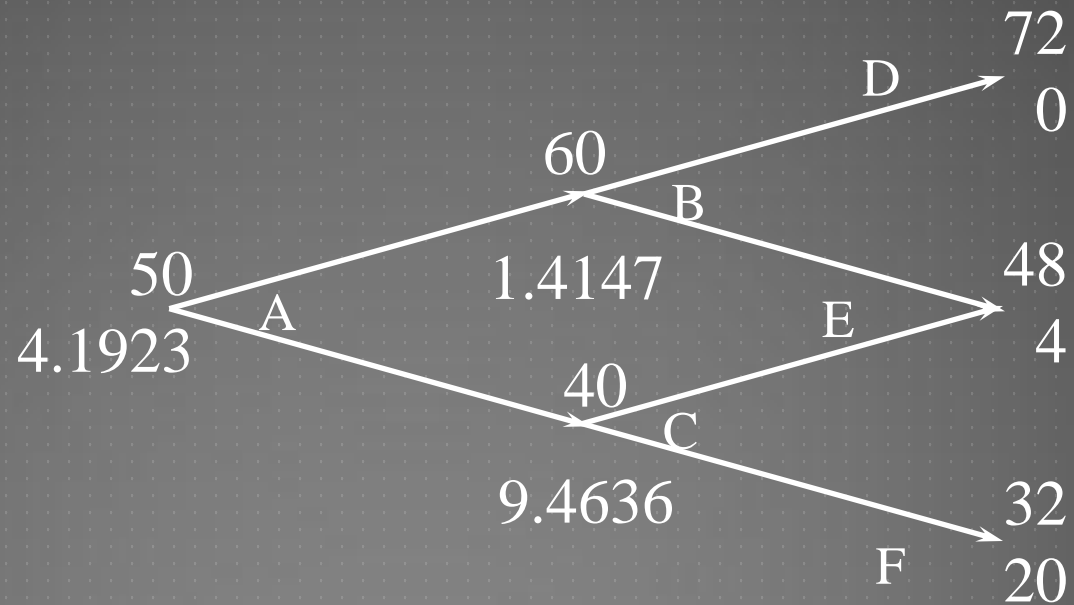
- ▶ Value at node A is

$$e^{-0.12 \times 0.25} (0.6523 \times 2.0257 + 0.3477 \times 0) = 1.2823$$

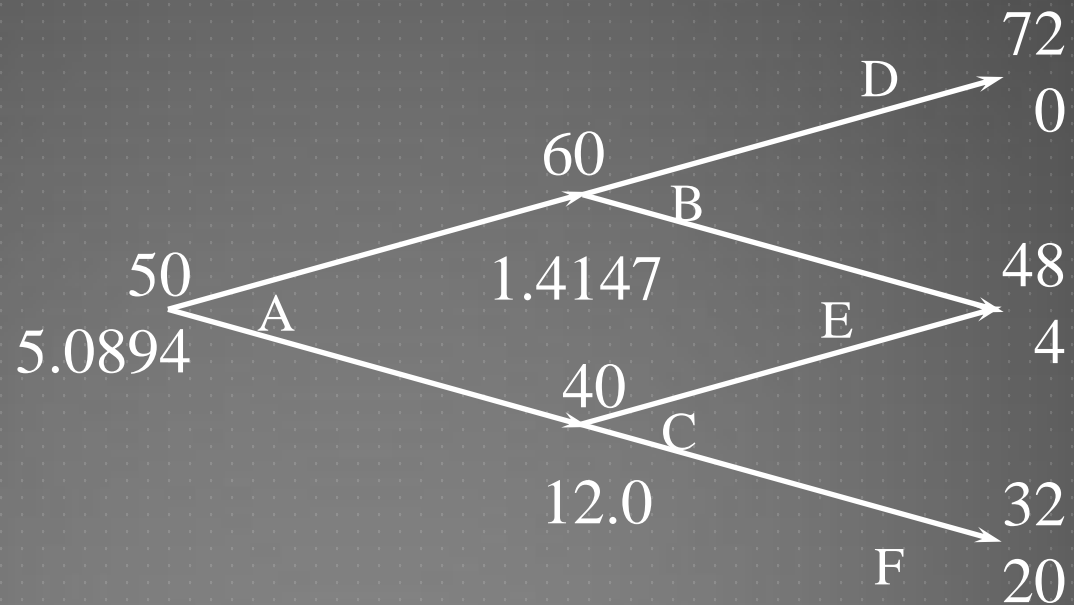
# A PUT OPTION EXAMPLE

$K = 52$ , time step = 1 yr

$r = 5\%$



# WHAT HAPPENS WHEN AN OPTION IS AMERICAN



# DELTA

- ▶ Delta ( $\Delta$ ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- ▶ The value of  $\Delta$  varies from node to node

# CHOOSING $U$ AND $D$

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where  $\sigma$  is the volatility and  $\Delta t$  is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein

# THE PROBABILITY OF AN UP MOVE

$$p = \frac{a - d}{u - d}$$

- $a = e^{r\Delta t}$  for a nondividend paying stock
- $a = e^{(r-q)\Delta t}$  for a stock index where  $q$  is the dividend yield on the index
- $a = e^{(r-r_f)\Delta t}$  for a currency where  $r_f$  is the foreign risk-free rate
- $a = 1$  for a futures contract

# THE BLACK-SCHOLES-MERTON MODEL



# THE STOCK PRICE ASSUMPTION

- ▶ Consider a stock whose price is  $S$
- ▶ In a short period of time of length  $\Delta t$ , the return on the stock is normally distributed:

$$\frac{\Delta S}{S} \approx \phi(\mu \Delta t, \sigma^2 \Delta t)$$

where  $\mu$  is expected return and  $\sigma$  is volatility

# THE LOGNORMAL PROPERTY

- ▶ It follows from this assumption that

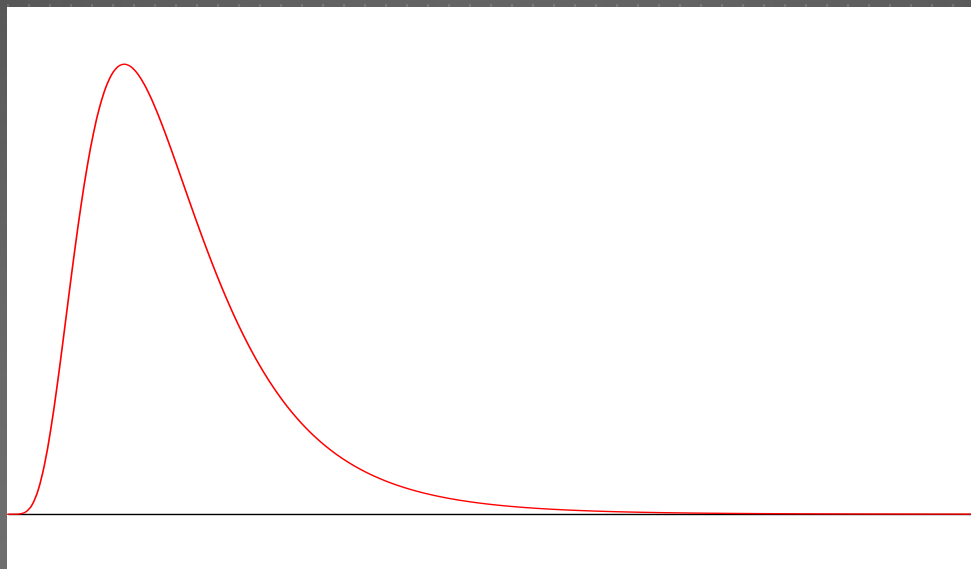
$$\ln S_T - \ln S_0 \approx \phi \left[ \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right]$$

or

$$\ln S_T \approx \phi \left[ \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right]$$

- ▶ Since the logarithm of  $S_T$  is normal,  $S_T$  is lognormally distributed

# THE LOGNORMAL DISTRIBUTION



$$E(S_T) = S_0 e^{\mu T}$$

$$\text{var}(S_T) = S_0^2 e^{2\mu T} (e^{\sigma^2 T} - 1)$$

# CONTINUOUSLY COMPOUNDED RETURN

If  $x$  is the continuously compounded return

$$S_T = S_0 e^{xT}$$

$$x = \frac{1}{T} \ln \frac{S_T}{S_0}$$

$$x \approx \phi \left( \mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T} \right)$$

# THE EXPECTED RETURN

- ▶ The expected value of the stock price is  $S_0 e^{\mu T}$
- ▶ The expected return on the stock is  $\mu - \sigma^2/2$  not  $\mu$

This is because

$$\ln[E(S_T / S_0)] \quad \text{and} \quad E[\ln(S_T / S_0)]$$

are not the same

# M AND $M - \Sigma^2/2$

Suppose we have daily data for a period of several months

- ▶  $\mu$  is the average of the returns in each day [ $=E(\Delta S/S)$ ]
- ▶  $\mu - \sigma^2/2$  is the expected return over the whole period covered by the data measured with continuous compounding (or daily compounding, which is almost the same)

# THE CONCEPTS UNDERLYING BLACK-SCHOLES

- ▶ The option price and the stock price depend on the same underlying source of uncertainty
- ▶ We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- ▶ The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- ▶ This leads to the Black-Scholes differential equation

# THE DERIVATION OF THE BLACK-SCHOLES DIFFERENTIAL EQUATION

$$\Delta S = \mu S \Delta t + \sigma S \Delta z$$

$$\Delta f = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta z$$

We set up a portfolio consisting of

–1: derivative

+  $\frac{\partial f}{\partial S}$ : shares

# THE DERIVATION OF THE BLACK-SCHOLES DIFFERENTIAL EQUATION CONTINUED

The value of the portfolio  $\Pi$  is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time  $\Delta t$  is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

# THE DERIVATION OF THE BLACK-SCHOLES DIFFERENTIAL EQUATION CONTINUED

The return on the portfolio must be the risk - free rate. Hence

$$\Delta\Pi = r \Pi\Delta t$$

We substitute for  $\Delta f$  and  $\Delta S$  in these equations to get the Black - Scholes differential equation :

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

# THE DIFFERENTIAL EQUATION

- ▶ Any security whose price is dependent on the stock price satisfies the differential equation
- ▶ The particular security being valued is determined by the boundary conditions of the differential equation
- ▶ In a forward contract the boundary condition is  $f = S - K$  when  $t = T$
- ▶ The solution to the equation is

$$f = S - K e^{-r(T-t)}$$

# THE BLACK-SCHOLES FORMULAS

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where  $d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

# THE $N(x)$ FUNCTION

- ▶  $N(x)$  is the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 is less than  $x$

# PROPERTIES OF BLACK-SCHOLES FORMULA

- ▶ As  $S_0$  becomes very large  $c$  tends to  $S_0 - Ke^{-rT}$  and  $p$  tends to zero
- ▶ As  $S_0$  becomes very small  $c$  tends to zero and  $p$  tends to  $Ke^{-rT} - S_0$

# RISK-NEUTRAL VALUATION

- ▶ The variable  $\mu$  does not appear in the Black-Scholes equation
- ▶ The equation is independent of all variables affected by risk preference
- ▶ The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- ▶ This leads to the principle of risk-neutral valuation

# APPLYING RISK-NEUTRAL VALUATION

1. Assume that the expected return from the stock price is the risk-free rate
2. Calculate the expected payoff from the option
3. Discount at the risk-free rate

# VALUING A FORWARD CONTRACT WITH RISK-NEUTRAL VALUATION

- ▶ Payoff is  $S_T - K$
- ▶ Expected payoff in a risk-neutral world is  $S_0 e^{rT} - K$
- ▶ Present value of expected payoff is

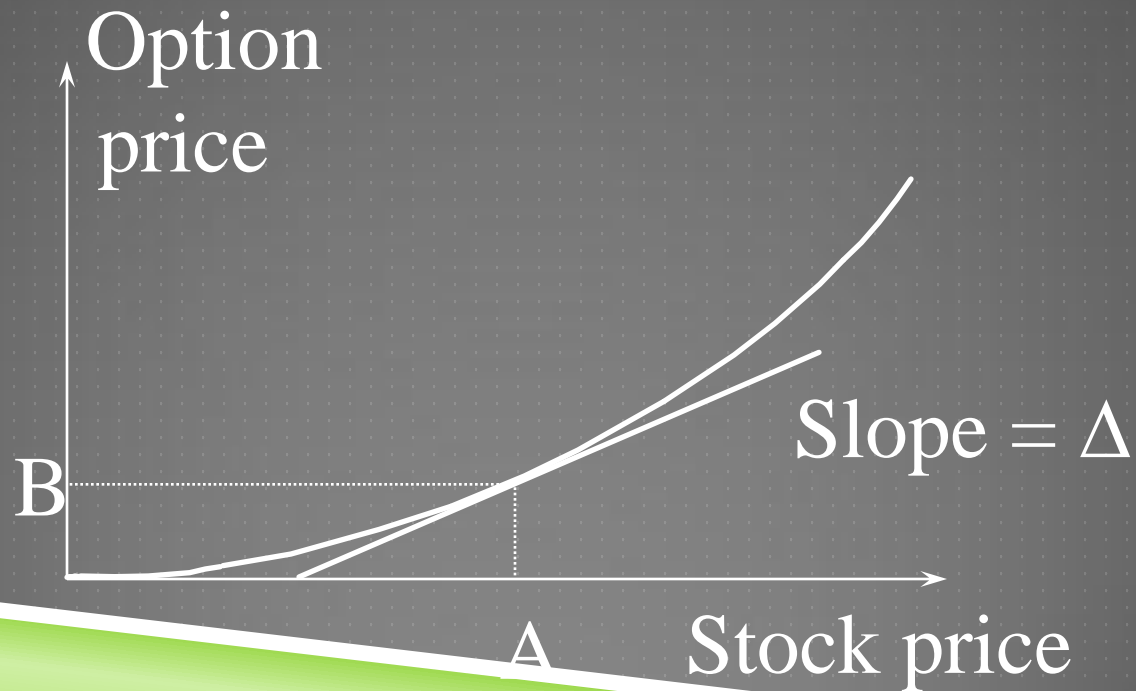
$$e^{-rT}[S_0 e^{rT} - K] = S_0 - K e^{-rT}$$

# IMPLIED VOLATILITY

- ▶ The implied volatility of an option is the volatility for which the Black-Scholes price equals the market price
- ▶ There is a one-to-one correspondence between prices and implied volatilities
- ▶ Traders and brokers often quote implied volatilities rather than dollar prices

# DELTA

- ▶ Delta ( $\Delta$ ) is the rate of change of the option price with respect to the underlying



# DELTA HEDGING

- ▶ This involves maintaining a delta neutral portfolio
- ▶ The delta of a European call on a non-dividend paying stock is  $N(d_1)$
- ▶ The delta of a European put on the stock is

$$N(d_1) - 1$$

# DELTA HEDGING CONTINUED

- ▶ The hedge position must be frequently rebalanced
- ▶ Delta hedging a written option involves a “buy high, sell low” trading rule

# DELTA

$$\Delta_c = \frac{\partial c}{\partial S} = e^{-r^*\tau} N(d_1) \quad (13.9)$$

► **Key concept:**

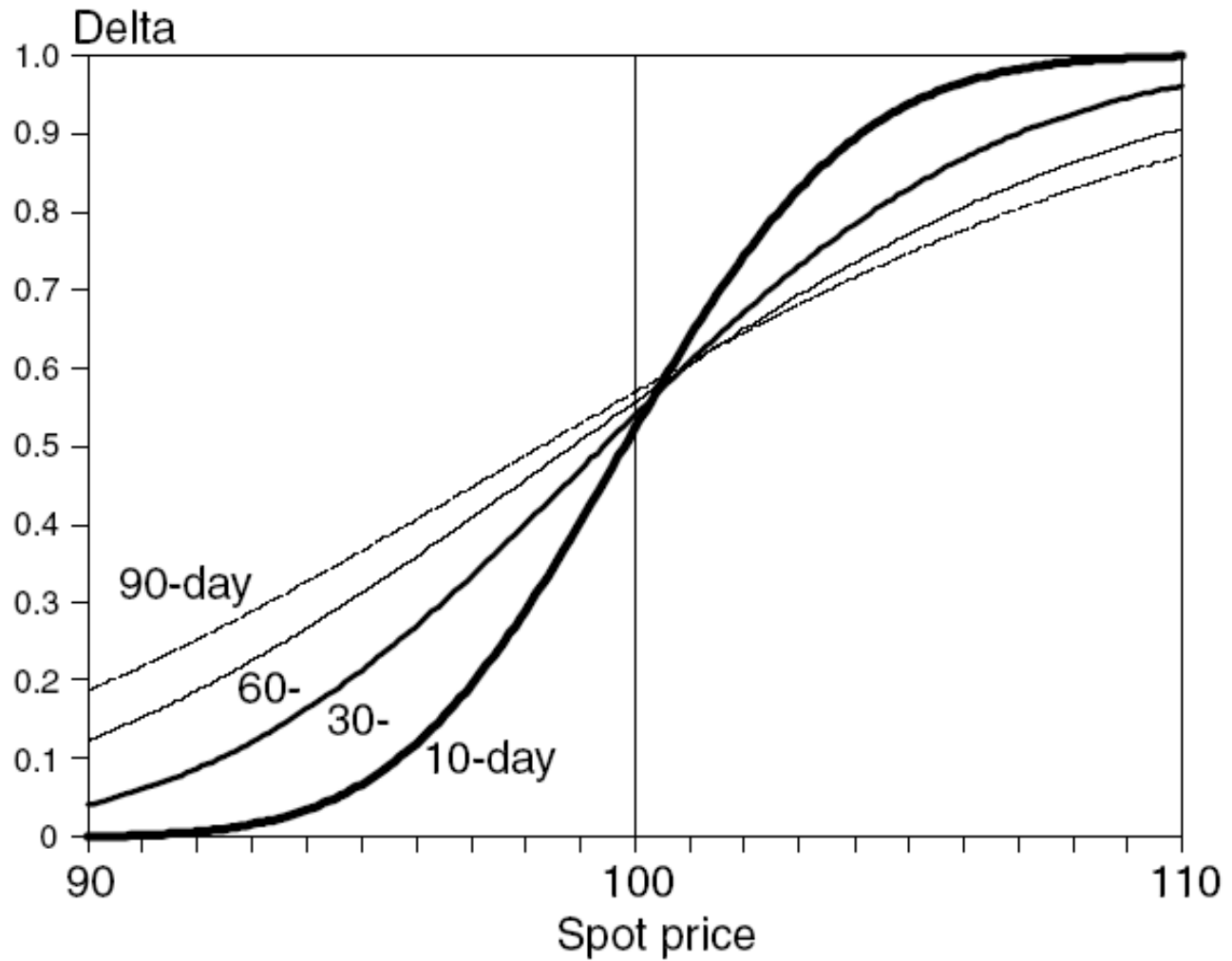
The delta of an at-the-money call option is close to 0.5. Delta moves to 1 as the call goes deep in-the-money. It moves to 0 as the call goes deep out-of-the-money

$$\Delta_p = \frac{\partial p}{\partial S} = e^{-r^*\tau} [N(d_1) - 1] \quad (13.10)$$

► **Key concept:**

The delta of an at-the-money put option is close to -0.5. Delta moves to -1 as the put goes deep in-the-money. It moves to 0 as the put goes deep out-of-the-money

**FIGURE 13-2 Option Delta**



# THETA

- Theta ( $\Theta$ ) of a derivative (or portfolio of derivatives) is the rate of change of the value with respect to the passage of time
- The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of a long option declines

# THETA

⊖ measures the sensitivity of an option to time

Unlike other factors the movement in remaining maturity is perfectly predictable →  
Time is not a risk factor

$$\Theta_c = \frac{\partial c}{\partial t} = -\frac{\partial c}{\partial \tau} = -\frac{Se^{-r^*\tau}\sigma\Phi(d_1)}{2\sqrt{\tau}} + r^*Se^{-r^*\tau}N(d_1) - rKe^{-r\tau}N(d_2) \quad (13.17)$$

$$\Theta_p = \frac{\partial p}{\partial t} = -\frac{\partial p}{\partial \tau} = -\frac{Se^{-r^*\tau}\sigma\Phi(d_1)}{2\sqrt{\tau}} - r^*Se^{-r^*\tau}N(-d_1) + rKe^{-r\tau}N(-d_2) \quad (13.18)$$

⊖ is generally negative for long positions in both calls and puts → the option loses value as time goes by

At-the-money options lose a lot of value when the maturity is near

For American options ⊖ is *always* negative → shorter-term American options are unambiguously less valuable than longer-term options

# GAMMA

- ▶  $\Gamma$  is the second derivative of the option price with respect to the price of the underlying asset

$$\Gamma = \frac{\partial^2 c}{\partial S^2} = \frac{e^{-r^* \tau} \Phi(d_1)}{S \sigma \sqrt{\tau}} \quad (13.11)$$

$\Gamma$  is derivative of  $\Delta$  with respect to  $S \rightarrow$  it measures how stable  $\Delta$  is

$\Gamma$  is driven by the “bell shape” of the normal density function

$\Gamma$  is identical for a call and put with identical characteristics

At-the-money options have the highest gamma  $\rightarrow \Delta$  moves very fast as  $S$  moves

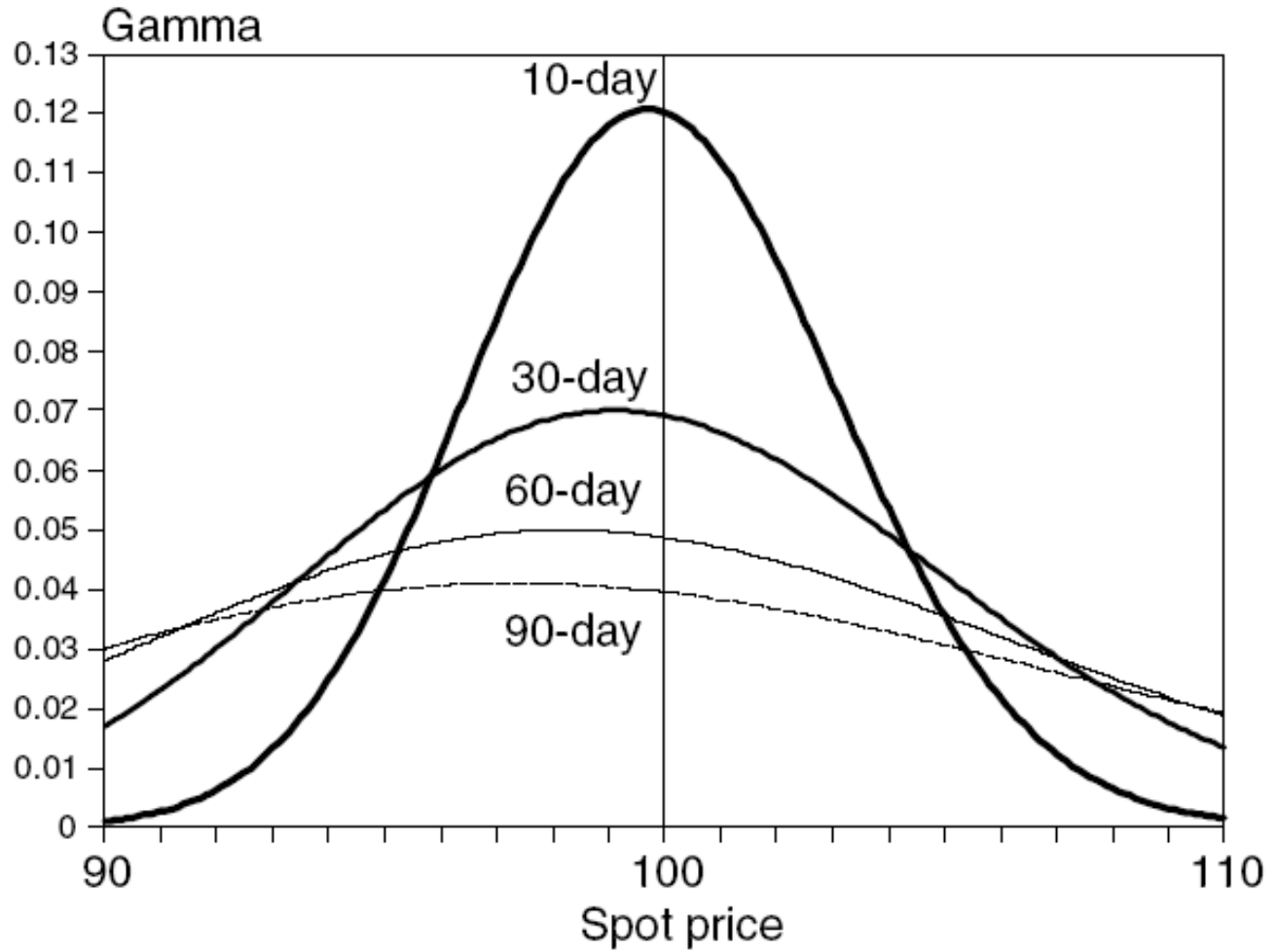
In-the-money options & out-of-the-money options have low gammas because their  $\Delta$  is constant, close to 1 or 0, respectively

As the maturity nears, the option gamma increases



Nonlinearities are most pronounced for short term at the money options

**FIGURE 13-3 Option Gamma**



# GAMMA

$\Gamma$  for options is similar to the concept of convexity for bonds

Bonds usually have positive convexity, whereas options can create positive or negative  $\Gamma$

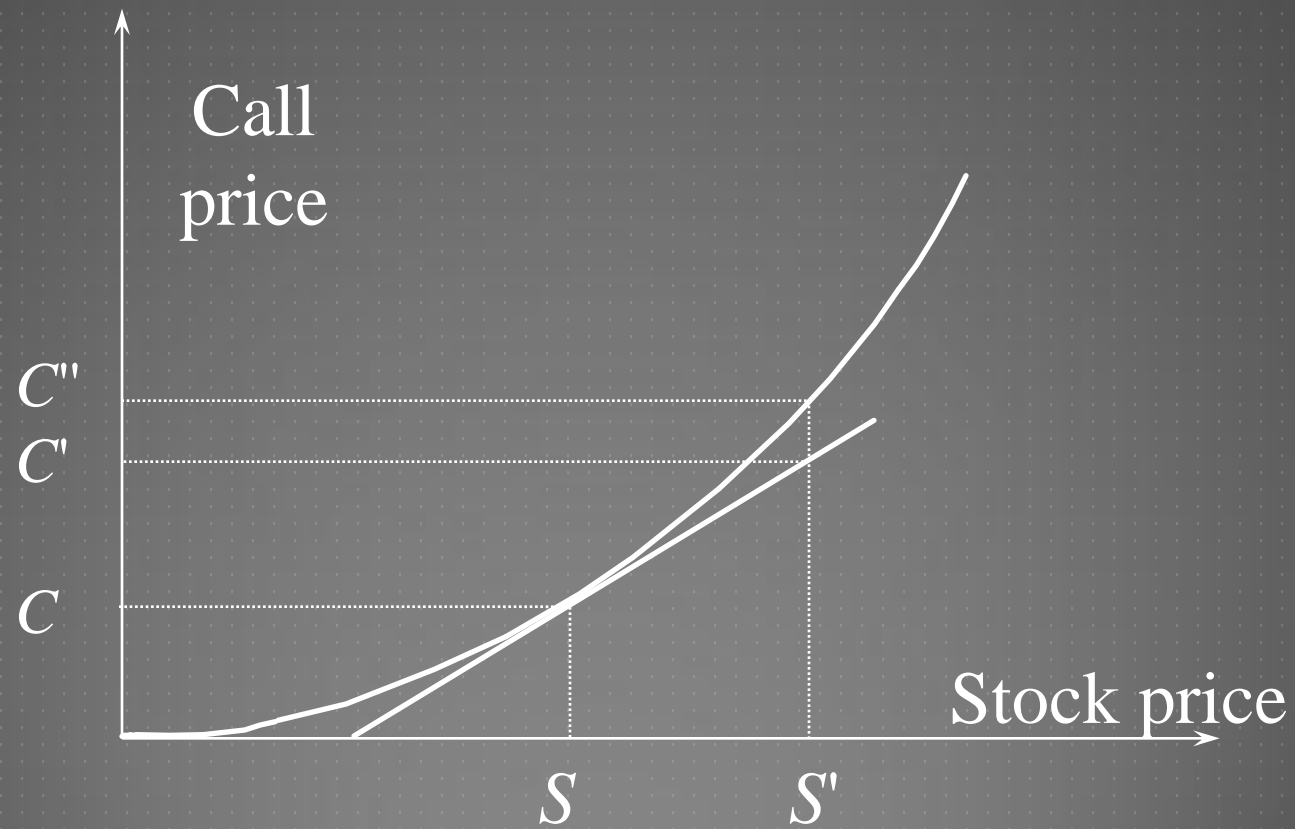
Positive convexity or  $\Gamma$  is beneficial  $\rightarrow$  the value of the asset drops more slowly and increase more quickly when the market risk factor moves

Negative convexity and  $\Gamma$  can be dangerous because it implies faster price falls and slower price increases

Long positions in options, whether calls or puts, create positive convexity

Short positions create negative convexity  $\rightarrow$  In exchange for assuming the harmful effect of this negative convexity, option sellers receive the premium

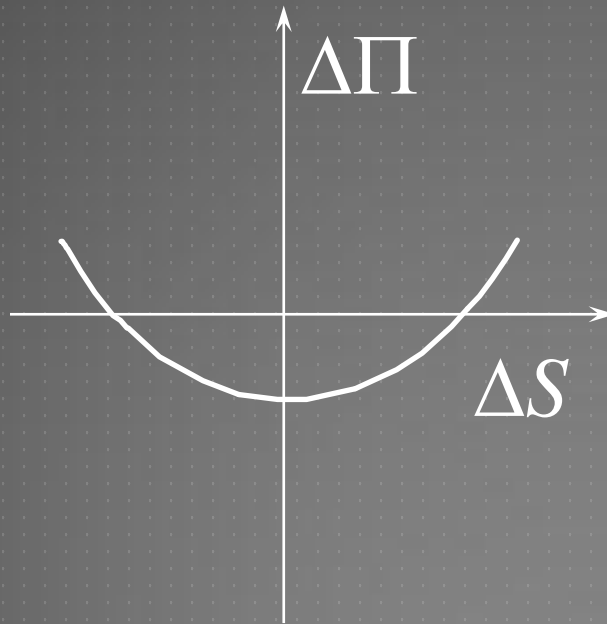
# GAMMA ADDRESSES DELTA HEDGING ERRORS CAUSED BY CURVATURE



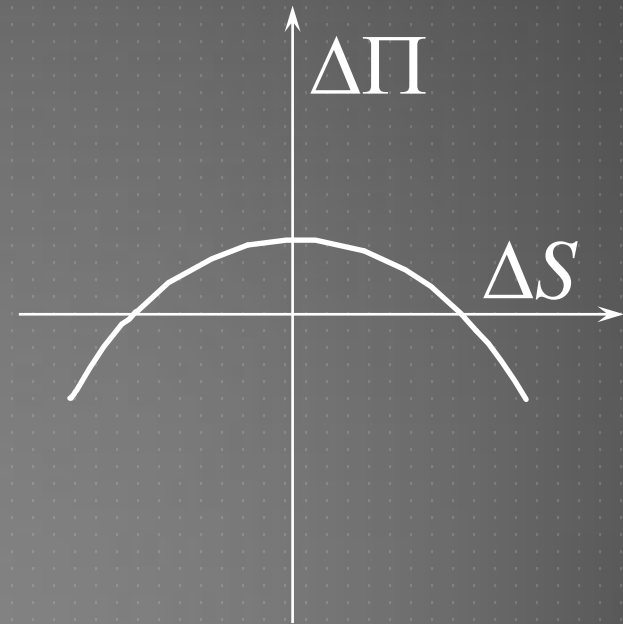
# INTERPRETATION OF GAMMA

- For a delta neutral portfolio,

$$\Delta\Pi \approx \Theta \Delta t + \frac{1}{2}\Gamma\Delta S^2$$



Positive Gamma



Negative Gamma

# RELATIONSHIP BETWEEN DELTA, GAMMA, AND THETA

For a portfolio of derivatives on a stock paying a continuous dividend yield at rate  $q$

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2\Gamma = r\Pi$$

# VEGA

- ▶ Vega ( $V$ ) is the rate of change of the value of a derivatives portfolio with respect to volatility
- ▶ Vega tends to be greatest for options that are close to the money

# VEGA

$\Lambda$  measures the sensitivity of an option to volatility

Volatility is very important in option pricing

$$\Lambda = \frac{\partial c}{\partial \sigma} = S e^{-r^* \tau} \sqrt{\tau} \Phi(d_1) \quad (13.12)$$

$\Lambda$  has the bell shape of the normal density function (similar to  $\Gamma$ )

$\Lambda$  is identical for calls and puts

$\Lambda > 0$  for long position in options

At-the money option are most sensitive to volatility

$\Lambda$  decreases with maturity

# MANAGING DELTA, GAMMA, & VEGA

- $\Delta$  can be changed by taking a position in the underlying
- ▶ To adjust  $\Gamma$  &  $v$  it is necessary to take a position in an option or other derivative

# RHO

- ▶ Rho is the rate of change of the value of a derivative with respect to the interest rate

# RHO

$\rho$  measures the sensitivity of an option to domestic interest rate  $r$

$$\rho_c = \frac{\partial c}{\partial r} = K e^{-r\tau} \tau N(d_2) \quad (13.13)$$

$$\rho_p = \frac{\partial p}{\partial r} = -K e^{-r\tau} \tau N(-d_2) \quad (13.14)$$

An increase in the rate of interest increases the value of the call  
→ the underlying asset grows at a higher rate, which increases the probability of exercising the call → in the limit, for an infinite interest rate, the probability of exercise is 1 and the call option is equivalent to the stock itself

The reasoning is opposite for a put option

# RHO\*

$\rho^*$  measures the sensitivity of an option to dividend yield (or foreign interest rate)  $r^*$

$$\rho_C^* = \frac{\partial c}{\partial r^*} = -Se^{-r^*\tau} \tau N(d_1) \quad (13.15)$$

$$\rho_P^* = \frac{\partial p}{\partial r^*} = Se^{-r^*\tau} \tau N(-d_1) \quad (13.16)$$

An increase in the dividend yield decreases the growth rate of the underlying asset  $\rightarrow$  the probability to exercise the call option is lower  $\rightarrow$  the value of the call option goes down

The reasoning is opposite for a put option

# HEDGING IN PRACTICE

- ▶ Traders usually ensure that their portfolios are delta-neutral at least once a day
- ▶ Whenever the opportunity arises, they improve gamma and vega
- ▶ As portfolio becomes larger hedging becomes less expensive

# SCENARIO ANALYSIS

A scenario analysis involves testing the effect on the value of a portfolio of different assumptions concerning asset prices and their volatilities

# FUTURES CONTRACT CAN BE USED FOR HEDGING

- ▶ The delta of a futures contract on an asset paying a yield at rate  $q$  is  $e^{(r-q)T}$  times the delta of a spot contract
- ▶ The position required in futures for delta hedging is therefore  $e^{-(r-q)T}$  times the position required in the corresponding spot contract

# HEDGING VS CREATION OF AN OPTION SYNTHETICALLY

- ▶ When we are hedging we take positions that offset  $\Delta, \Gamma, V$ , etc.
- ▶ When we create an option synthetically we take positions that match  $\Delta, \Gamma, \& V$

# PORTFOLIO INSURANCE

- ▶ In October of 1987 many portfolio managers attempted to create a put option on a portfolio synthetically
- ▶ This involves initially selling enough of the portfolio (or of index futures) to match the  $\Delta$  of the put option

# PORTFOLIO INSURANCE CONTINUED

- ▶ As the value of the portfolio increases, the  $\Delta$  of the put becomes less negative and some of the original portfolio is repurchased
- ▶ As the value of the portfolio decreases, the  $\Delta$  of the put becomes more negative and more of the portfolio must be sold

# PORTFOLIO INSURANCE CONTINUED

The strategy did not work well on October 19, 1987...

# ITO'S LEMMA

- ▶ If we know the stochastic process followed by  $x$ , Ito's lemma tells us the stochastic process followed by some function  $G(x, t)$
- ▶ Derivatives are functions of the underlying asset,  $x$ , and time,  $t$

Implication:

if we know the process for  $x$  (the underlying asset), by applying Ito's lemma, we can compute the process for  $G(x, t)$

# ITO'S LEMMA

A Taylor's series expansion of  $G(x, t)$  gives

$$\delta G = \frac{\partial G}{\partial x} \delta x + \frac{\partial G}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \delta x^2 + \frac{\partial^2 G}{\partial x \partial t} \delta x \delta t + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} \delta t^2 + \dots$$

In ordinary calculus we have

$$\delta G = \frac{\partial G}{\partial x} \delta x + \frac{\partial G}{\partial t} \delta t$$

In stochastic calculus this becomes

$$\delta G = \frac{\partial G}{\partial x} \delta x + \frac{\partial G}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \delta x^2$$

because  $\delta x$  is function of  $\sqrt{\delta t}$

# ITO'S LEMMA

Assume that

$$\delta x = a \delta t + b \varepsilon \sqrt{\delta t}$$

Then ignoring terms of higher order than  $\delta t$ , the Taylor expansion gives us

$$\delta G = \frac{\partial G}{\partial x} \delta x + \frac{\partial G}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \varepsilon^2 \delta t$$

Since  $\varepsilon \approx \phi(0,1) \implies E(\varepsilon) = 0$ ,  $V[\varepsilon] = E(\varepsilon^2) - [E(\varepsilon)]^2 = 1$ ,  $E(\varepsilon^2) = 1$

It follows that  $E(\varepsilon^2 \delta t) = \delta t$

The variance of  $\delta t$  is proportional to  $\delta t^2$  and can be ignored  $\implies$

$$\delta G = \frac{\partial G}{\partial x} \delta x + \frac{\partial G}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \delta t$$

# ITO'S LEMMA

Substituting

$$\delta x = a \delta t + b \delta z$$

into

$$\delta G = \frac{\partial G}{\partial x} \delta x + \frac{\partial G}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \delta t$$

We obtain

$$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz$$

This is Ito's Lemma

# ITO'S LEMMA: APPLICATION

The stock price process is  $dS = \mu S dt + \sigma S dz$

For a function  $G$  of  $S$  and  $t$

$$dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz$$

1. The forward price of a stock for a contract maturing at time  $T$

$$G = S e^{r(T-t)}$$

$$dG = (\mu - r)G dt + \sigma G dz$$

2.  $G = \ln S$

$$dG = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dz$$

# ITO'S LEMMA: OPTION PRICING

Assume the usual GBM model for the asset

$$dS = \mu S dt + \sigma S dz$$

Applying Ito's lemma to the option price  $f(S, t)$ , we have

$$df = \left( \frac{\partial f}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 + \frac{\partial f}{\partial \tau} \right) dt + \frac{\partial f}{\partial S} \sigma S dz$$

$$df = (\Delta \mu S + \frac{1}{2} \Gamma \sigma^2 S^2 + \Theta) dt + (\Delta \sigma S) dz$$

The 1st term in  $dt$  is the trend while the second term is the stochastic component

# OPTION PRICING AND THE GREEKS

Let's go back to the example where we construct a balanced risk-free portfolio composed by the underlying asset,  $S$ , and the option,  $f$

$$\Pi = f - \Delta S$$

$$d\Pi = df - \Delta dS$$

$$d\Pi = (\Delta\mu S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt + (\Delta\sigma S)dz - \Delta[\mu Sdt + \sigma Sdz]$$

$$d\Pi = (\frac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt$$

This simplification is extremely important  $\rightarrow$  the terms involving  $dz$  cancel each other out  $\rightarrow$  The portfolio has been immunized against this source of risk

Also the terms in  $\mu S$  also cancel each other out  $\rightarrow$  the trend of the underlying asset,  $\mu$  does not appear in the Black-Scholes formula

# OPTION PRICING AND THE GREEKS

The portfolio  $\Pi = f - \Delta S$  has no risk  $\rightarrow$  to avoid arbitrage this portfolio must earn the risk free rate

$$d\Pi = [r\Pi]dt = r(f - \Delta S)dt$$

if the underlying asset earns a dividend yield, we have

$$d\Pi = [r\Pi]dt + y\Delta Sdt = r(f - \Delta S)dt + y\Delta Sdt \quad (13.25)$$

We also know that  $d\Pi = (0.5\Gamma\sigma^2S^2 + \Theta)dt$  (13.23)

By setting (13.23) = (13.25), we have:

$$(r - y)\Delta S + 0.5\Gamma\sigma^2S^2 + \Theta = rf$$

This is a the PDE  $\rightarrow$  the solution of this equation is the BS formula for call options

**TABLE 13-1 Derivatives for a European Call**

**Parameters:**  $S = \$100$ ,  $\sigma = 20\%$ ,  $r = 5\%$ ,  $y = 3\%$ ,  $\tau = 3$  months

	Variable	Unit	Strike			Worst Loss	
			$K = 90$	$K = 100$	$K = 110$	Variable	Loss
$c$		Dollars	\$11.02	\$4.22	\$1.05		
		Change per:					
$\Delta$	Spot price	dollar	0.868	0.536	0.197	-\$2.08	-\$1.114
$\Gamma$	Spot price	dollar	0.020	0.039	0.028	4.33	\$0.084
$\Lambda$	Volatility	(% pa)	0.103	0.198	0.139	-2.5	-\$0.495
$\rho$	Interest rate	(% pa)	0.191	0.124	0.047	-0.10	-\$0.013
$\rho^*$	Asset yield	(% pa)	-0.220	-0.135	-0.049	0.10	-\$0.014
$\Theta$	Time	day	-0.014	-0.024	-0.016		

# TABLE 13.1

95% confidence level: 1.645

$$dS = -1.645 \times 20\% \times \$100 / \sqrt{252} = -\$2.07$$

$$\Delta \times dS = -\$1.111 \quad (25\% \text{ of the option value})$$

$$dS^2 = 2.07^2 = 4.30$$

$$0.5\Gamma \times dS^2 = 0.5 \times 0.039 \times 4.30 = 0.084$$

This is a gain because gamma is positive, but much smaller than the first order effect

$$\sigma: 1.5\% \rightarrow \text{worse daily movement: } -1.645 \times 1.5 = -2.5\%$$

r: annual volatility of change in interest rate of 1%

$$\text{Worse daily loss: } -1.645 \times 1 / \sqrt{252} = -0.10\%$$

Most of the risk originates from  $S$  and in particular from  $\Delta$

# DYNAMIC HEDGING

The BS formula is based on a replicating portfolio: a call option is equivalent to holding a fraction ( $\Delta$ ) of the underlying asset,  $S$

$\Delta$  is the first derivative of the option price with respect to the underlying asset,  $S \rightarrow \Delta$  is not constant over time  $\rightarrow$  to replicate a call option when the price is moving implies changing the fraction ( $\Delta$ ) of the underlying asset owned  $\rightarrow$  dynamic hedging

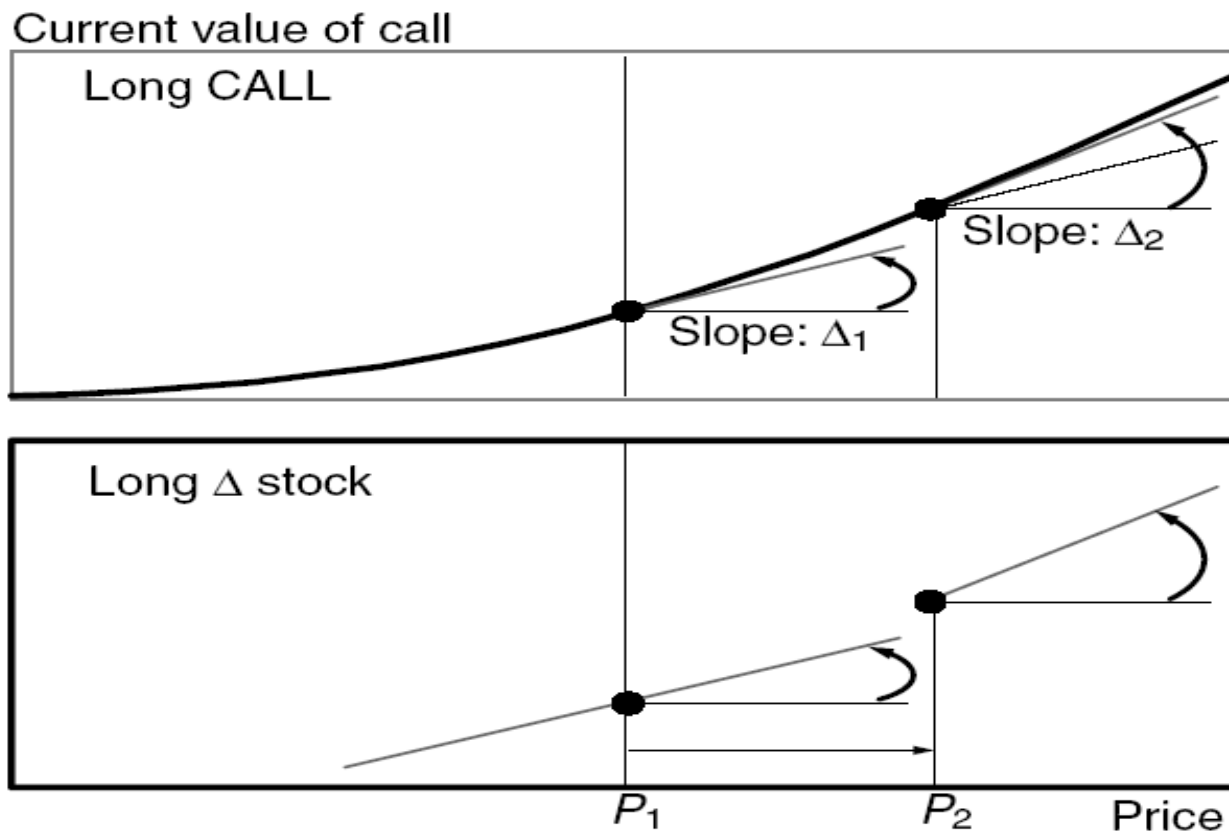
As the stock price increases from  $P_1$  to  $P_2$ , the slope of the option curve ( $\Delta$ ) increases  $\rightarrow$  the option can be replicated by a larger position in the underlying asset

When the stock price decreases, the size of the position is cut

Dynamic hedging  $\rightarrow$  buy more of the asset as its price goes up and, conversely, sells it after a fall (the same is also valid for a long position in a put Option)

In contrast, short positions in calls and puts imply the opposite pattern  $\rightarrow$  Dynamic hedging implies selling more of the asset after its price has gone up

**FIGURE 13-8 Dynamic Replication of a Call Option**



# INTERESTING QUESTION

- ▶ Shall we rebalance our portfolio every time the price move?

Be careful!!!

# DISTRIBUTION OF OPTION PAYOFFS

Unlike linear derivatives such as forwards and futures, payoffs on options are asymmetric

This is not necessarily because of the distribution of the underlying factor, which is often symmetric

Long positions in options, whether calls or puts, have positive gamma, positive skewness, or long right tails

Short positions in options are short gamma and hence have negative skewness or long left tails

# VAR FOR OPTIONS

Assumption: Normal distribution ( $\alpha$  confidence level,  $\alpha = 1.645$  for 95% confidence level)

VaR for the underlying asset:  $VaR = \alpha S \sigma(dS/S)$

Linear VaR for option:  $VaR_1(dc) = \Delta VaR(dS)$

Quadratic VaR for option:  $VaR_1(dc) = \Delta VaR(dS) - 0.5 \Gamma VaR(dS)$

# VAR FOR OPTIONS

- ▶ In computing VaR for options it is important the nonlinearity of the option payoffs
- ▶ The degree of nonlinearity depends on the horizon → With a VaR horizon of two weeks, the range of possible values for  $S$  is quite narrow → If  $S$  follows a normal distribution, the option value will be approximately normal

If the VAR horizon is set at two months, the nonlinearities in the exposure combine with the greater range of price movements create a heavily skewed distribution

So for plain-vanilla options, the linear approximation may be adequate as long as the VaR horizon is kept short

For more exotic options, or longer VAR horizons, the risk manager needs to account for nonlinearities