MONTE CARLO VALUATION

MONTE CARLO VALUATION

- Simulation of future stock prices and using these simulated prices to compute the discounted expected payoff of an option
  - Draw random numbers from an appropriate distribution
  - Use risk-neutral probabilities, and therefore risk-free discount rate
  - Distribution of payoffs a byproduct

MONTE CARLO VALUATION (CONT'D)

- Simulation of future stock prices and using these simulated prices to compute the discounted expected payoff of an option (cont'd)
  - Pricing of asset claims and assessing the risks of the asset
  - Control variate method increases conversion speed
  - Incorporate jumps by mixing Poisson and lognormal variables
  - Simulated correlated random variables can be created using Cholesky decomposition

MONTE CARLO SIMULATION AND OPTIONS

When used to value European stock options, Monte Carlo simulation involves the following steps:

1. Simulate 1 path for the stock price in a risk neutral world
2. Calculate the payoff from the stock option
3. Repeat steps 1 and 2 many times to get many sample payoff
4. Calculate mean payoff
5. Discount mean payoff at risk free rate to get an estimate of the value of the option

COMPUTING THE OPTION PRICE AS A DISCOUNTED EXPECTED VALUE

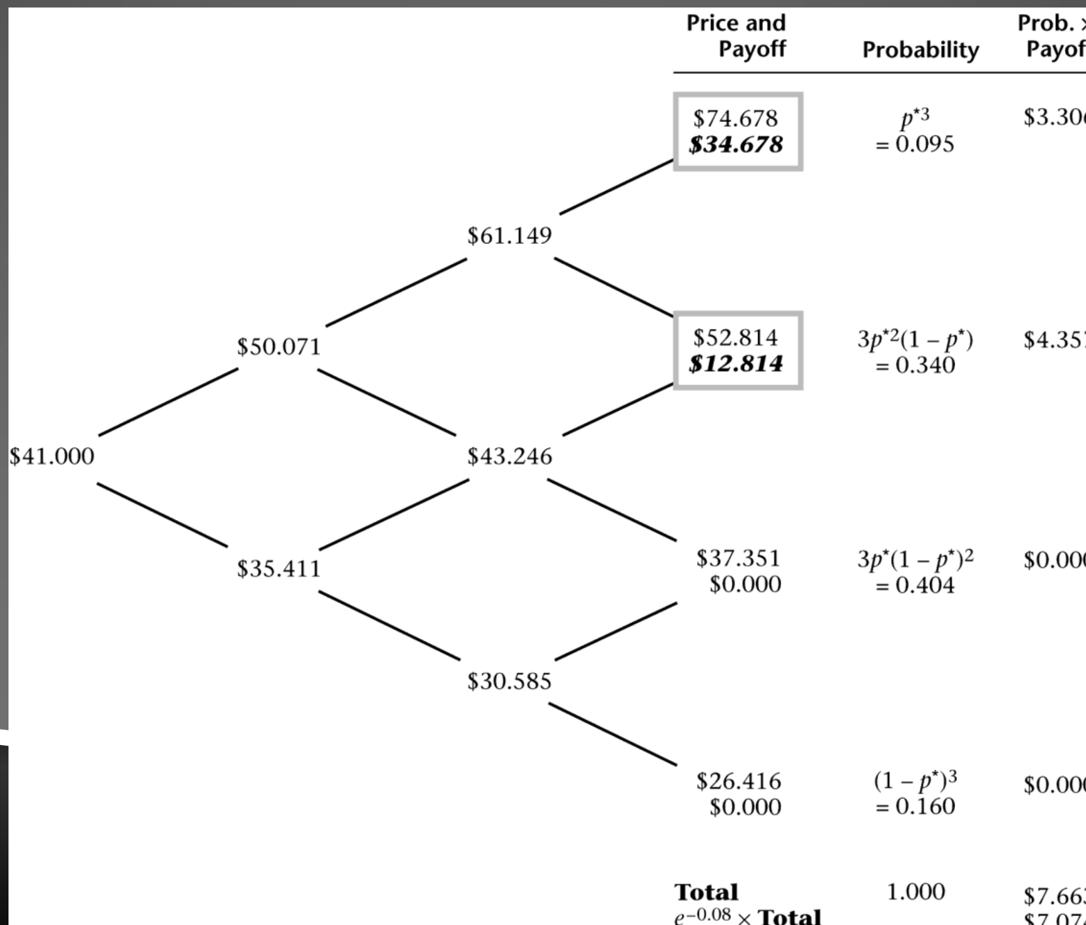
- Assume a stock price distribution 3 months from now
- For each stock price drawn from the distribution compute the payoff of a call option (repeat many times)
- Take the expectation of the resulting option payoff distribution using the risk-neutral probability p^*

COMPUTING THE OPTION PRICE AS A DISCOUNTED EXPECTED VALUE (CONT'D)

- Discount the average payoff at the risk-free rate of return
- In a binomial setting, if there are *n* binomial steps, and *i* down moves of the stock price, the European Call price is

$$\text{European Call Price} = e^{-rT}\sum_{i=1}^{n} max[0, Su^{n-i}d^{i} - K](p^{*})^{n-1}(1-p^{*})^{i}\frac{n!}{(n-i)!i!}$$

COMPUTING THE OPTION PRICE AS A DISCOUNTED EXPECTED VALUE (CONT'D)



COMPUTING THE OPTION PRICE AS A DISCOUNTED EXPECTED VALUE (CONT'D)

TABLE 19.1 Computation of option price using expected value calculation and true probabilities. The stock price tree and parameters are the same as in Figure 11.4. The column entitled "Discount Rates along Path" reports the node-specific true annualized continuously compounded discount rates from that figure. "Discount Rate for Path" is the compound annualized discount rate for the entire path. "Prob. of Path" is the probability that the particular path will occur, computed using the true probability of an up move (52.46%). The last column is the probability times the payoff, discounted at the continuously compounded rate for the path.

Path	Discount Rates along Path			Discount Rate for Path	Prob. of Path	Payoff ($)	Discounted ($) (Prob. × Payoff)
uuu	35.7%	32.3%	26.9%	31.64%	0.1444	34.678	3.649
uud	35.7%	32.3%	26.9%	31.64%	0.1308	12.814	1.222
udu	35.7%	32.3%	49.5%	39.18%	0.1308	12.814	1.133
duu	35.7%	49.5%	49.5%	44.91%	0.1308	12.814	1.070
udd	—	—	—	—	—	0	0
dud	—	—	—	—	—	0	0
ddu	—	—	—	—	—	0	0
ddd	—	—	—	—	—	0	0
					Sum		7.074

COMPUTING RANDOM NUMBERS

- There are several ways for generating random numbers
  - Use "RAND" function in Excel to generate random numbers between 0 and 1 from a uniform distribution $U(0,1)$
  - To generate random numbers (approximately) from a standard normal distribution $N(0,1)$, sum 12 uniform (0,1) random variables and subtract 6
  - To generate random numbers from any distribution D (for which an inverse cumulative distribution D^{-1} can be computed),
    - generate a random number x from $U(0,1)$
    - find z such that $D(z) = x$, *i.e.*, $D^{-1}(x) = z$
    - repeat

SIMULATING LOGNORMAL STOCK PRICES

- Recall that if $Z \sim N(0,1)$, a lognormal stock price is

$$S_t = S_0 e^{(\alpha - 0.5\sigma^2)t + \sigma\sqrt{t}Z}$$

- Randomly draw a set of standard normal Z's and substitute the results into the equation above. The resulting S_t's will be lognormally distributed random variables at time *t*.
- To simulate the path taken by *S* (which is useful in valuing path-dependent options) split *t* into *n* intervals of length *h*

$$S_h = S_0 e^{(\alpha - 0.5\sigma^2)h + \sigma\sqrt{h}Z(1)}$$

$$S_{2h} = S_h e^{(\alpha - 0.5\sigma^2)h + \sigma\sqrt{h}Z(2)}$$

$$S_{nh} = S_{(n-1)h} e^{(\alpha - 0.5\sigma^2)h + \sigma\sqrt{h}Z(n)}$$

MONTE CARLO VALUATION

- If $V(S_t,t)$ is the option payoff at time t, then the time-0 Monte Carlo price $V(S_0,0)$ is

$$V(S_0,0)=\frac{1}{n}e^{-rT}\sum_{i=1}^{n}V(S_T^i,T)$$

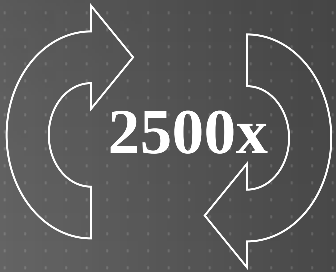
  - where $S_T^1, \ldots, S_T^n$ are n randomly drawn time-T stock prices

- For the case of a call option $V(S_T^i,T) = max\,(0, S_T^i-K)$

MONTE CARLO VALUATION (CONT'D)

- Example: Value a 3-month European call where the S_0=\$40, K=\$40, r=8%, and σ=30%

$$S_{3\,months} = S_0 e^{(0.08-0.3^2/2)\times 0.25+0.3\sqrt{0.25}Z}$$



- For each stock price, compute

Option payoff = $max(0, S_{3\ months} - \$40)$

- Average the resulting payoffs
- Discount the average back 3 months at the risk-free rate

\$2.804 versus \$2.78 Black-Scholes price

MONTE CARLO VALUATION (CONT'D)

- Monte Carlo valuation of American options is not as easy
- Monte Carlo valuation is inefficient
  - 500 observations σ=\$0.180 6.5%
  - 2500 observations σ=\$0.080 2.9%
  - 21,000 observations σ=\$0.028 1.0%
- Monte Carlo valuation of options is especially useful when
  - Number of random elements in the valuation problem is two great to permit direct numerical valuation
  - Underlying variables are distributed in such a way that direct solutions are difficult
  - The payoff depends on the path of underlying asset price

MONTE CARLO VALUATION (CONT'D)

- Monte Carlo valuation of Asian options
  - The payoff is based on the average price over the life of the option

$$S_1 = 40e^{(r-0.5\sigma^2)t/3+\sigma\sqrt{t/3Z}(1)}$$

$$S_2 = 40e^{(r-0.5\sigma^2)t/3+\sigma\sqrt{t/3Z}(2)}$$

$$S_3 = 40e^{(r-0.5\sigma^2)t/3+\sigma\sqrt{t/3Z}(3)}$$

  - The value of the Asian option is computed as

$$C_{asian} = e^{-rt}E(\max[(S_1 + S_1 + S_1)/3 - K, 0])$$

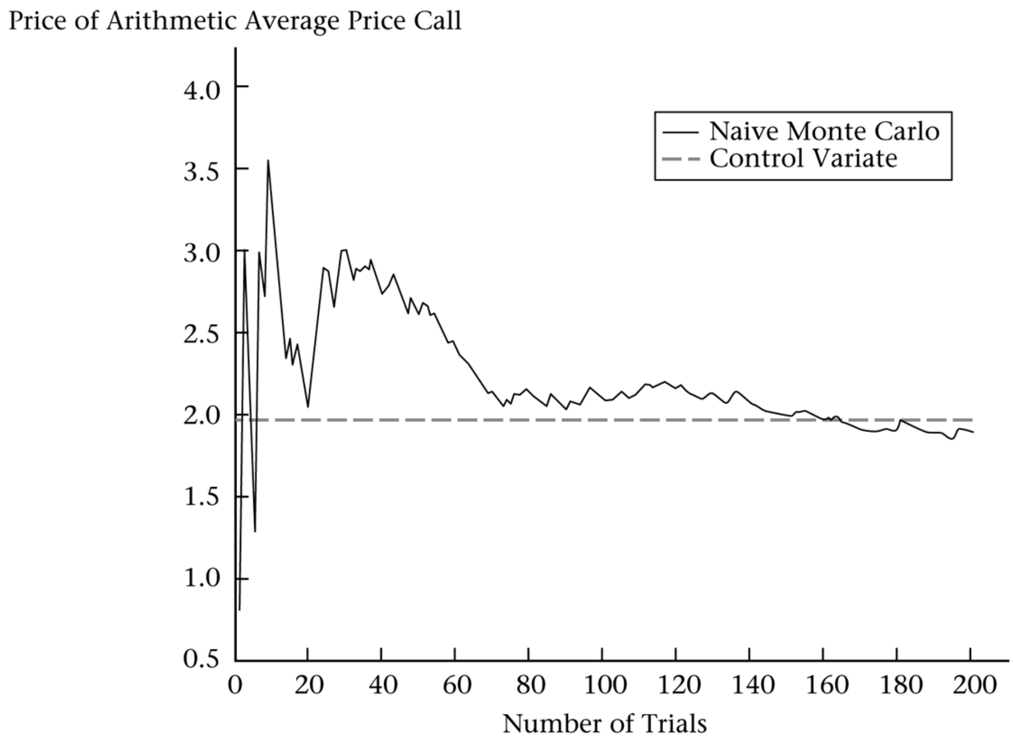
MONTE CARLO VALUATION (CONT'D)

TABLE 19.3 Prices of arithmetic average-price Asian options estimated using Monte Carlo and exact prices of geometric average price options. Assumes option has 3 months to expiration and average is computed using equal intervals over the period. Each price is computed using 10,000 trials, assuming $S = \$40$, $K = \$40$, $\sigma = 30\%$, $r = 8\%$, $T = 0.25$, $\delta = 0$. In each row, the same random numbers were used to compute both the geometric and arithmetic average price options. σ_n is the standard deviation of the estimated prices, divided by $\sqrt{10{,}000}$.

Number of Averages	Monte Carlo Prices ($)		Exact	
	Arithmetic	Geometric	Geometric Price ($)	σ_n
1	2.79	2.79	2.78	0.0408
3	2.03	1.99	1.94	0.0291
5	1.78	1.74	1.77	0.0259
10	1.70	1.66	1.65	0.0241
20	1.66	1.61	1.59	0.0231
40	1.63	1.58	1.56	0.0226

EFFICIENT MONTE CARLO VALUATION

FIGURE 19.3



Comparison of "naive" Monte Carlo estimate of arithmetic average option price with control variate method. Graph depicts first 200 simulations for an option with $S = \$40$, $K = \$40$, $\sigma = 0.3$, $r = 0.08$, $T = 0.25$, $\delta = 0$, and the final price computed with three averages.

EFFICIENT MONTE CARLO VALUATION (CONT'D)

- Control variate method
  - Estimate the error on each trial by using the price of an option that has a pricing formula.
  - Example: use errors from geometric Asian option to correct the estimate for the arithmetic Asian option price
- Antithetic variate method
  - For every draw also obtain the opposite and equally likely realizations to reduce variance of the estimate
- Stratified sampling
  - Treat each number as a random draw from each percentile of the uniform distribution
- Other methods
  - Importance sampling, low discrepancy sequences

THE POISSON DISTRIBUTION

- A discrete probability distribution that counts the number of events that occur over a period of time
  - λh is the probability that one event occurs over the short interval *h*
  - Over the time period *t* the probability that the event occurs exactly *m* times is given by

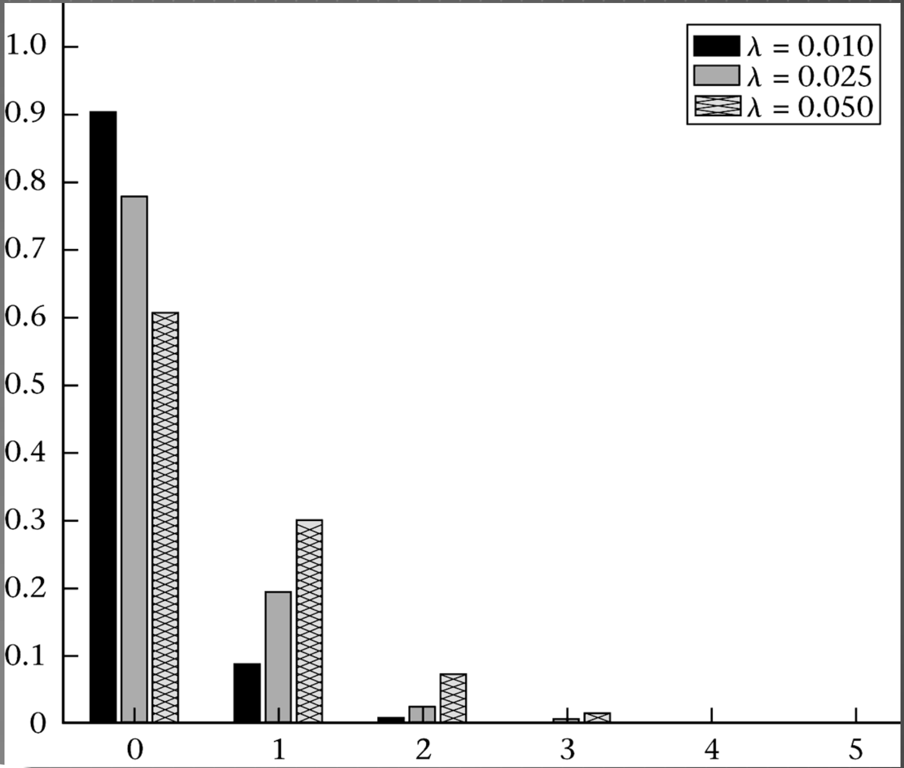
$$p(m,\lambda t) = \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

  - The cumulative Poisson distribution (the probability that there are *m* or fewer events from 0 to *t*) is

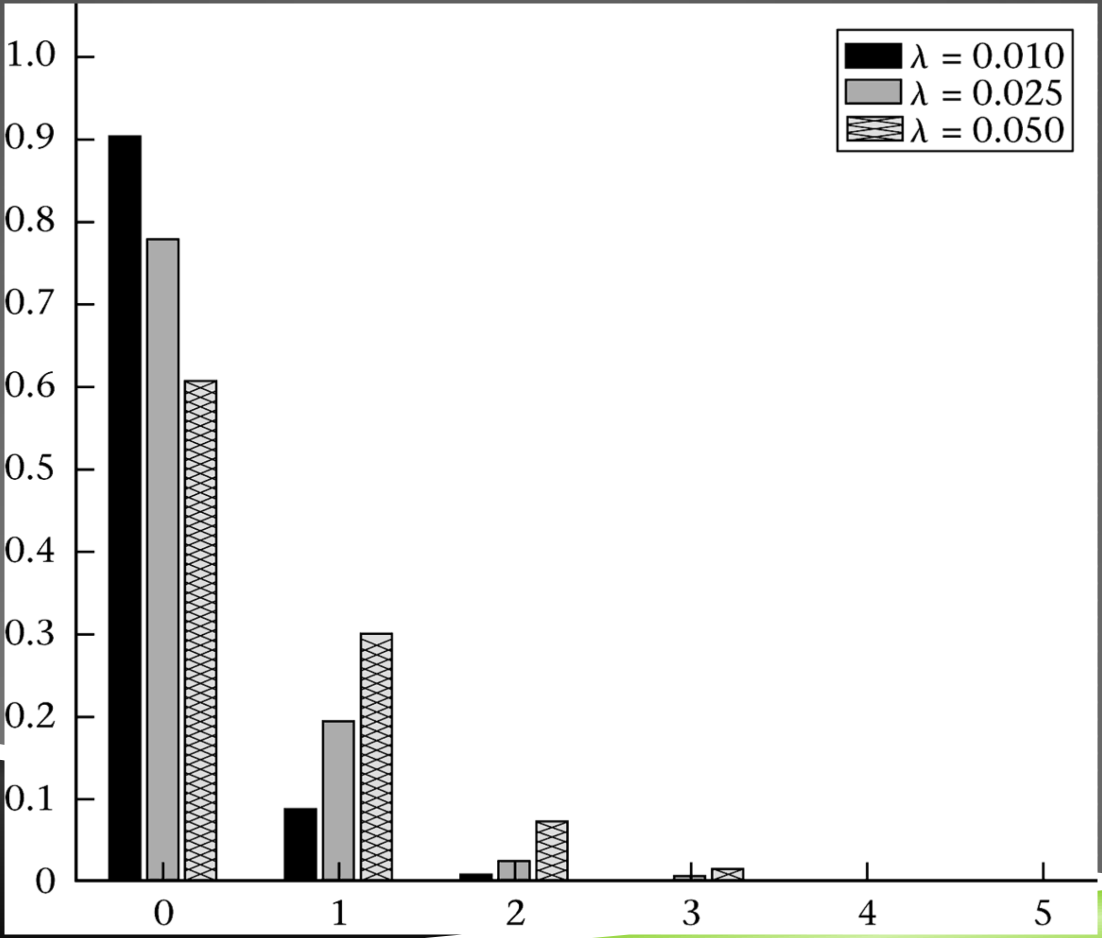
$$P(m,\lambda t) = Prob(x \le m; \lambda t) = \sum_{i=0}^{m} \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

THE POISSON DISTRIBUTION (CONT'D)

- The mean of the Poisson distribution is λt
- Given an expected number of events, the Poisson distribution gives the probability of seeing a particular number of events over a given time



THE POISSON DISTRIBUTION (CONT'D)



SIMULATING JUMPS WITH THE POISSON DISTRIBUTION

- Stock prices sometimes move ("jump") more than what one would expect to see under lognormal distribution
- The expression for lognormal stock price with *m* jumps is

$$\hat{S}_{t+h} = \hat{S}_t e^{(\alpha-\delta-\lambda k-0.5\sigma^2)t+\sigma\sqrt{t}Z} e^{m(\alpha_j-0.5\sigma_j)+\sigma_j\sum_{i=0}^{m} W_i}$$

where α_J and σ_J are the mean and standard deviation of the jump and Z and W_i are random standard normal variables

- To simulate this stock price at time *t+h* select
  - A standard normal *Z*
  - Number of jumps *m* from the Poisson distribution
  - *m* draws, *W*(*i*), *i*= 1, … , *m*, from the standard normal distribution

SIMULATING JUMPS WITH THE POISSON DISTRIBUTION (CONT'D)