

EXOTIC OPTIONS

A decorative graphic at the bottom of the slide consisting of a green trapezoidal shape pointing downwards, which is partially overlapped by a yellow trapezoidal shape pointing upwards. Both shapes have a white border.

EXOTIC OPTIONS

- ▶ Nonstandard options
- ▶ Exotic options solve particular business problems that an ordinary option cannot
- ▶ They are constructed by tweaking ordinary options in minor ways

EXOTIC OPTIONS (CONT'D)

▶ Relevant questions

- ▶ How does the exotic payoff compare to ordinary option payoff?
- ▶ Can the exotic option be approximated by a portfolio of other options?
- ▶ Is the exotic option cheap or expensive relative to standard options?
- ▶ What is the rationale for the use of the exotic option?
- ▶ How easily can the exotic option be hedged?

TYPES OF EXOTICS

- ▶ Package
- ▶ Nonstandard American options
- ▶ Forward start options
- ▶ Compound options
- ▶ Chooser options
- ▶ Barrier options
- ▶ Binary options
- ▶ Lookback options
- ▶ Shout options
- ▶ Asian options
- ▶ Options to exchange one asset for another
- ▶ Options involving several assets
- ▶ Volatility and Variance swaps

PACKAGES (PAGE 555)

- ▶ Portfolios of standard options
- ▶ Examples from Chapter 10: bull spreads, bear spreads, straddles, etc
- ▶ Often structured to have zero cost
- ▶ One popular package is a range forward contract

NON-STANDARD AMERICAN OPTIONS

(PAGE 556)

- ▶ Exercisable only on specific dates (Bermudans)
- ▶ Early exercise allowed during only part of life (initial “lock out” period)
- ▶ Strike price changes over the life (warrants, convertibles)

FORWARD START OPTIONS (PAGE 556)

- ▶ Option starts at a future time, T_1
- ▶ Implicit in employee stock option plans
- ▶ Often structured so that strike price equals asset price at time T_1

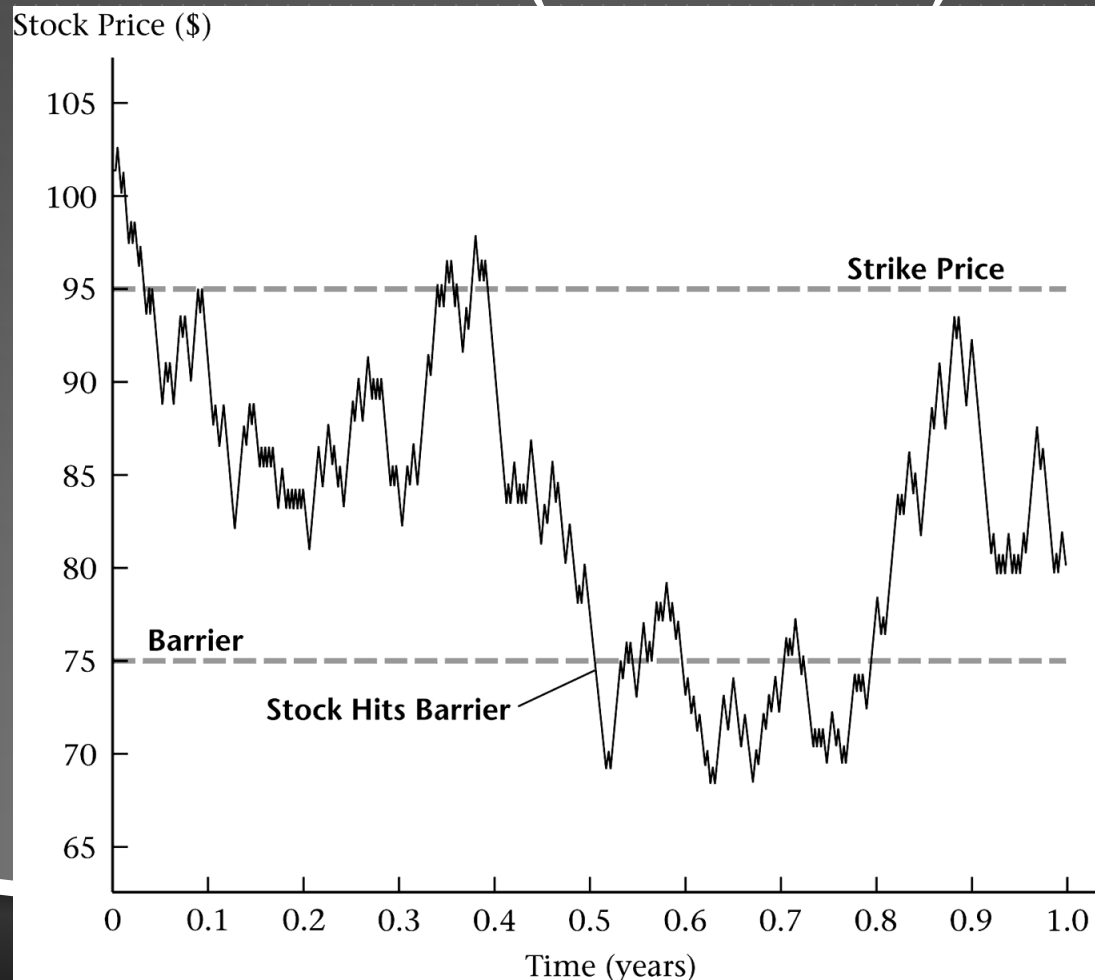
BARRIER OPTIONS

- ▶ The payoff depends on whether over the option life the underlying price reaches a specified level, called the *barrier*.
 - ▶ path dependent
- ▶ Barrier puts and calls either come into existence or go out of existence the first time the asset price reaches the barrier. If they are in existence at expiration, they are equivalent to ordinary puts and calls.
- ▶ Since barrier puts and calls never pay more than standard puts and calls, they are no more expensive than standard puts and calls.

BARRIER OPTIONS (CONT'D)

- ▶ Barrier puts and calls
 - ▶ Knock-out options: *go out of* existence if the asset price reaches the barrier.
 - ▶ down-and-out: has to *fall* to reach the barrier
 - ▶ up-and-out: has to *rise* to reach the barrier
 - ▶ Knock-in options: *come into* existence if the asset price
 - ▶ down-and-in: has to *fall* to reach the barrier
 - ▶ up-and-in: has to *rise* to reach the barrier
 - ▶ Rebate options: make a fixed payment if the asset price
 - ▶ down rebates: has to *fall* to reach the barrier
 - ▶ up rebates: has to *rise* to reach the barrier
 - ▶ Barrier options are less valuable than otherwise identical ordinary options

BARRIER OPTIONS (CONT'D)



BARRIER OPTIONS (CONT'D)

TABLE 14.4

Premiums of standard, down-and-in, and up-and-out currency put options with strikes K . The column headed "standard" contains prices of ordinary put options. Assumes $x_0 = 0.9$, $\sigma = 0.1$, $r_{\$} = 0.06$, $r_{\text{€}} = 0.03$, and $t = 0.5$.

Strike (\$)	Standard (\$)	Down-and-In Barrier (\$)		Up-and-Out Barrier (\$)		
		0.8000	0.8500	0.9500	1.0000	1.0500
$K = 0.8$	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007
$K = 0.9$	0.0188	0.0066	0.0167	0.0174	0.0188	0.0188
$K = 1.0$	0.0870	0.0134	0.0501	0.0633	0.0847	0.0869

PARITY RELATIONS

$$C = C_{ui} + C_{uo}$$

$$C = C_{di} + C_{do}$$

$$P = P_{ui} + P_{uo}$$

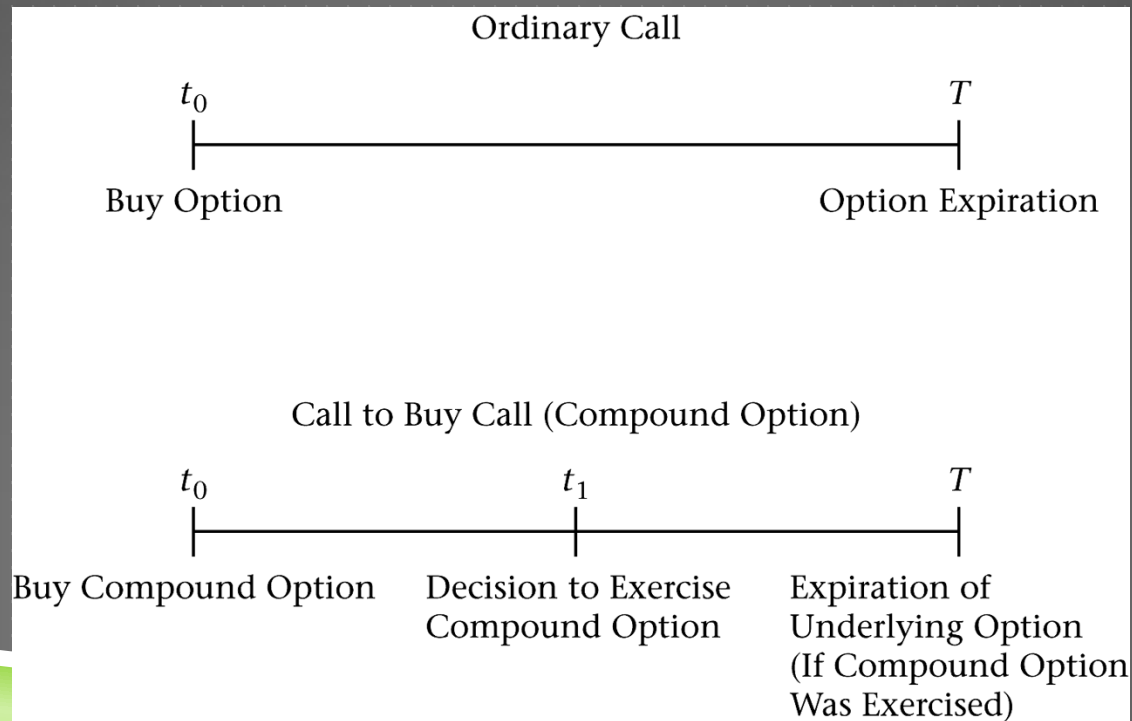
$$P = P_{di} + P_{do}$$

COMPOUND OPTION (PAGE 557)

- ▶ Option to buy or sell an option
 - ▶ Call on call
 - ▶ Put on call
 - ▶ Call on put
 - ▶ Put on put
- ▶ Can be valued analytically
- ▶ Price is quite low compared with a regular option

COMPOUND OPTIONS

- ▶ An option to buy an option



CHOOSE OPTION “AS YOU LIKE IT”

(PAGE 558)

- ▶ Option starts at time 0, matures at T_2
- ▶ At T_1 ($0 < T_1 < T_2$) buyer chooses whether it is a put or call
- ▶ This is a package!

CHOOSE OPTION AS A PACKAGE

At time T_1 the value is $\max(c, p)$

From put - call parity

$$p = c + e^{-r(T_2 - T_1)} K - S_1 e^{-q(T_2 - T_1)}$$

The value at time T_1 is therefore

$$c + e^{-q(T_2 - T_1)} \max(0, Ke^{-(r-q)(T_2 - T_1)} - S_1)$$

This is a call maturing at time T_2 plus
a put maturing at time T_1

BINARY OPTIONS (PAGE 561)

- ▶ Cash-or-nothing: pays Q if $S_T > K$, otherwise pays nothing.
 - ▶ Value = $e^{-rT} Q N(d_2)$
- ▶ Asset-or-nothing: pays S_T if $S_T > K$, otherwise pays nothing.
 - ▶ Value = $S_0 e^{-qT} N(d_1)$

DECOMPOSITION OF A CALL OPTION

Long Asset-or-Nothing option

Short Cash-or-Nothing option where payoff is K

$$\text{Value} = S_0 e^{-qT} N(d_1) - e^{-rT} KN(d_2)$$

LOOKBACK OPTIONS (PAGE 561-63)

- ▶ Floating lookback call pays $S_T - S_{\min}$ at time T (Allows buyer to buy stock at lowest observed price in some interval of time)
- ▶ Floating lookback put pays $S_{\max} - S_T$ at time T
(Allows buyer to sell stock at highest observed price in some interval of time)
- ▶ Fixed lookback call pays $\max(S_{\max} - K, 0)$
- ▶ Fixed lookback put pays $\max(K - S_{\min}, 0)$
- ▶ Analytic valuation for all types

SHOUT OPTIONS (PAGE 563-64)

- ▶ Buyer can 'shout' once during option life
- ▶ Final payoff is either
 - ▶ Usual option payoff, $\max(S_T - K, 0)$, or
 - ▶ Intrinsic value at time of shout, $S_\tau - K$
- ▶ Payoff: $\max(S_T - S_\tau, 0) + S_\tau - K$
- ▶ Similar to lookback option but cheaper
- ▶ How can a binomial tree be used to value a shout option?

ASIAN OPTIONS

- ▶ The payoff of an Asian option is based on the average price over some period of time
 - ▶ path-dependent
- ▶ Situations when Asian options are useful
 - ▶ When a business cares about the average exchange rate over time
 - ▶ When a single price at a point in time might be subject to manipulation
 - ▶ When price swings are frequent due to thin markets

ASIAN OPTIONS (CONT'D)

- ▶ Example
 - ▶ The exercise of the conversion option in convertible bonds is based on the stock price over a 20-day period at the end of the bond's life
- ▶ Asian options are less valuable than otherwise identical ordinary options

ASIAN OPTIONS (CONT'D)

- ▶ There are eight (2^3) basic kinds of Asian options:
 - ▶ Put or call
 - ▶ Geometric or arithmetic average
 - ▶ Average asset price is used in place of underlying price or strike
- ▶ Arithmetic versus geometric average:
 - ▶ Suppose we record the stock price every h periods from $t=0$ to $t=T$
 - ▶ Arithmetic average: Geometric average:

$$A(T) = \frac{1}{N} \sum_{i=1}^N S_{ih}$$

$$G(T) = (S_h \times S_{2h} \times \cdots \times S_{Nh})^{1/N}$$

ASIAN OPTIONS (CONT'D)

- ▶ Average used as the asset price: Average price option
 - ▶ Geometric average price call = $\max [0, G(T) - K]$
 - ▶ Geometric average price put = $\max [0, K - G(T)]$
- ▶ Average used as the strike price: Average strike option
 - ▶ Geometric average strike call = $\max [0, S_T - G(T)]$
 - ▶ Geometric average strike put = $\max [0, G(T) - S_T]$

ASIAN OPTIONS (CONT'D)

- ▶ All four options above could also be computed using arithmetic average instead of geometric average
- ▶ Simple pricing formulas exist for geometric average options but not for arithmetic average options

ASIAN OPTIONS (CONT'D)

► Comparing Asian options

TABLE 14.1

Premiums of at-the-money geometric average price and geometric average strike calls and puts, for different numbers of prices averaged, N . The case $N = 1$ for the average price options is equivalent to Black-Scholes values. Assumes $S = \$40$, $K = \$40$, $r = 0.08$, $\sigma = 0.3$, $\delta = 0$, and $t = 1$.

N	Average Price (\$)		Average Strike (\$)	
	Call	Put	Call	Put
1	6.285	3.209	0.000	0.000
2	4.708	2.645	2.225	1.213
3	4.209	2.445	2.748	1.436
5	3.819	2.281	3.148	1.610
10	3.530	2.155	3.440	1.740
50	3.302	2.052	3.668	1.843
1000	3.248	2.027	3.722	1.868
∞	3.246	2.026	3.725	1.869

ASIAN OPTIONS (CONT'D)

▶ XYZ's hedging problem

- ▶ XYZ has monthly revenue of €100m, and costs in dollars
- ▶ x is the dollar price of a euro
- ▶ In one year, the converted amount in dollars is

- ▶ Ignoring interest what needs to be hedged is

$$€100\text{m} \times \sum_{i=1}^{12} x_i e^{r(12-i)/12}$$

$$\sum_{i=1}^{12} x_i = 12 \times \left(\frac{\sum_{i=1}^{12} x_i}{12} \right)$$

ASIAN OPTIONS (CONT'D)

- ▶ A solution for XYZ

- ▶ If XYZ receives euros and its costs are fixed in dollars, profits are reduced if the euro depreciates. An Asian put option that puts a floor K , on the average rate received

$$\max \left(0, K - \frac{1}{12} \sum_{i=1}^{12} x_i \right)$$

- ▶ For example, if we wanted to guarantee an average exchange rate of \$0.90 per euro, we would set $K = \$0.90$. If the average $x < 0.90$, we would be paid the difference.

ASIAN OPTIONS (CONT'D)

► Alternative solutions for XYZ's hedging problem

TABLE 14.2

Comparison of costs for alternative hedging strategies for XYZ. The price in the second row is the sum of premiums for puts expiring after 1 month, 2 months, and so forth, out to 12 months. The first, third, and fourth rows premiums are calculated assuming 1 year to maturity, and then multiplied by 12. Assumes the current exchange rate is \$0.9/€, option strikes are 0.9, $r_{\$} = 6\%$, $r_{\text{€}} = 3\%$, dollar/euro volatility is 10%.

Hedge Instrument	Premium (\$)
Put Option Expiring in 1 Year	0.2753
Strip of Monthly Put Options	0.2178
Geometric Average Price Put	0.1796
Arithmetic Average Price Put	0.1764

GAP OPTIONS

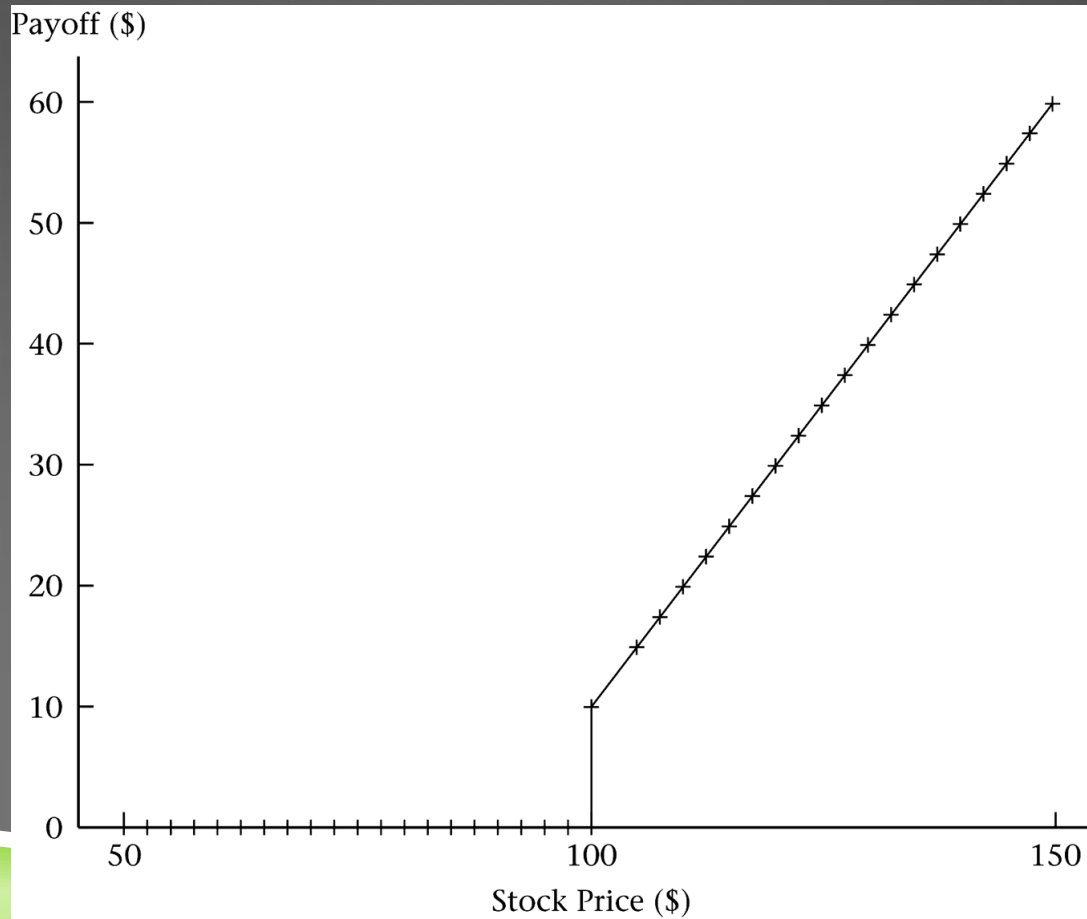
- ▶ A call option pays $S-K$ when $S > K$.
- ▶ A gap call option pays $S - K_1$ when $S > K_2$
- ▶ The value of a gap call is

$$C(S, K_1, K_2, \sigma, r, t, \delta) = Se^{-\delta t} N(d_1) - K_1 e^{-rt} N(d_2)$$

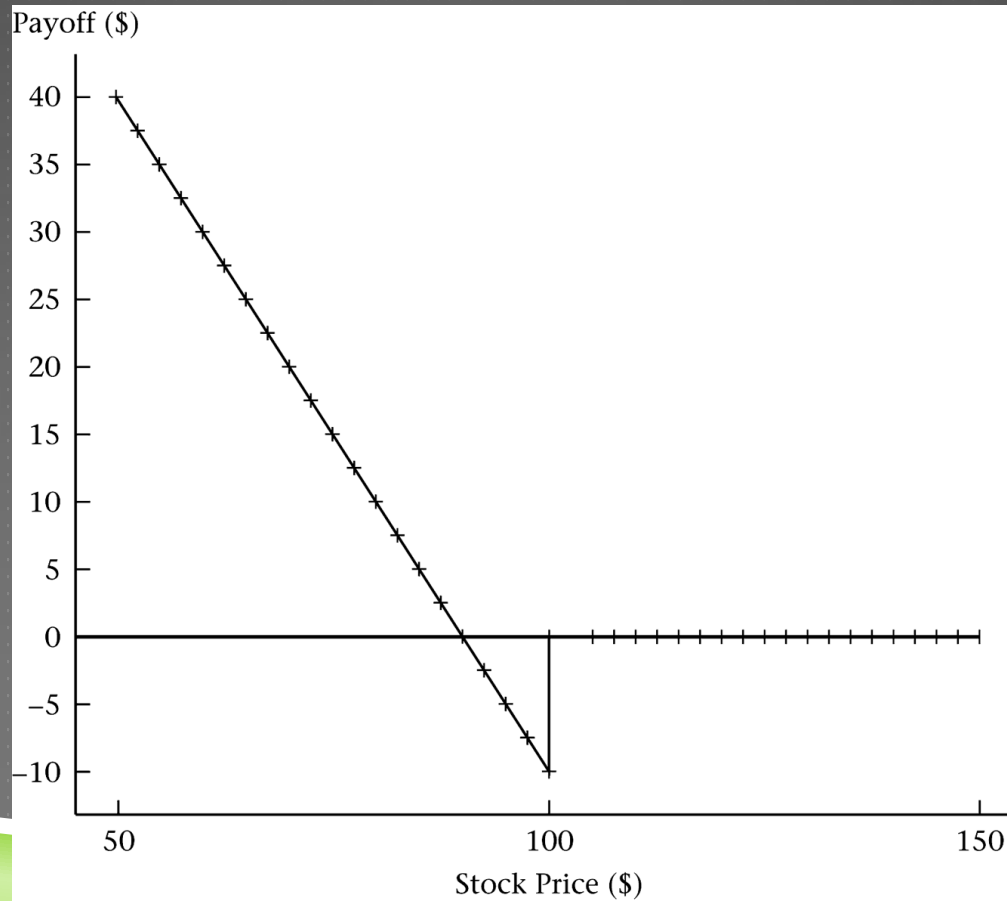
where

$$d_1 = \frac{\ln(S/K_2) + (r - \delta + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{t}$$

GAP OPTIONS (CONT'D)



GAP OPTIONS (CONT'D)



GAP OPTIONS (CONT'D)

TABLE 14.5

Premiums of ordinary and gap put options with strikes K_1 and payment triggers K_2 . Assumes $x_0 = 0.9$, $\sigma = 0.1$, $r_{\$} = 0.06$, $r_{\text{€}} = 0.03$, and $t = 0.5$.

Strike (K_1) (\$)	Put (\$)	Payment Trigger (K_2) (\$)		
		0.8	0.9	1.0
0.8000	0.0007	0.0007	-0.0229	-0.0888
0.9000	0.0188	0.0039	0.0188	-0.0009
1.0000	0.0870	0.0070	0.0605	0.0870

EXCHANGE OPTIONS

- ▶ Pays off only if the underlying asset outperforms some other asset (benchmark) out-performance option
- ▶ The value of a European exchange \rightarrow call is

$$C(S, K, \sigma, r, t, \delta) = S e^{-\delta_S t} N(d_1) - K e^{-\delta_K t} N(d_2)$$

$$d_1 = \frac{\ln(S e^{-\delta_S t} / K e^{-\delta_K t}) + \frac{1}{2} \sigma^2 t}{\sigma \sqrt{t}}, \quad d_2 = d_1 - \sigma \sqrt{t}$$

and

$$\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K}$$

BASKET OPTIONS (PAGE 567)

- ▶ A basket option is an option to buy or sell a portfolio of assets
- ▶ This can be valued by calculating the first two moments of the value of the basket and then assuming it is lognormal

VOLATILITY AND VARIANCE SWAPS

- ▶ Agreement to exchange the realized volatility between time 0 and time T for a prespecified fixed volatility with both being multiplied by a prespecified principal
- ▶ Variance swap is agreement to exchange the realized variance rate between time 0 and time T for a prespecified fixed variance rate with both being multiplied by a prespecified principal
- ▶ Daily return is assumed to be zero in calculating the volatility or variance rate

VARIANCE SWAPS (PAGE 568-69)

- ▶ The (risk-neutral) expected variance rate between times 0 and T can be calculated from the prices of European call and put options with different strikes and maturity T
- ▶ Variance swaps can therefore be valued analytically if enough options trade
- ▶ For a volatility swap it is necessary to use the approximate relation

$$\hat{E}(\bar{\sigma}) = \sqrt{\hat{E}(\bar{V})} \left\{ 1 - \frac{1}{8} \left[\frac{\text{var}(\bar{V})}{\hat{E}(\bar{V})^2} \right] \right\}$$

VIX INDEX (PAGE 570)

- ▶ The expected value of the variance of the S&P 500 over 30 days is calculated from the CBOE market prices of European put and call options on the S&P 500
- ▶ This is then multiplied by $365/30$ and the VIX index is set equal to the square root of the result

HOW DIFFICULT IS IT TO HEDGE EXOTIC OPTIONS?

- ▶ In some cases exotic options are easier to hedge than the corresponding vanilla options (e.g., Asian options)
- ▶ In other cases they are more difficult to hedge (e.g., barrier options)

STATIC OPTIONS REPLICATION

(SECTION 24.14, PAGE 570)

- ▶ This involves approximately replicating an exotic option with a portfolio of vanilla options
- ▶ Underlying principle: if we match the value of an exotic option on some boundary, we have matched it at all interior points of the boundary
- ▶ Static options replication can be contrasted with dynamic options replication where we have to trade continuously to match the option

EXAMPLE

- ▶ A 9-month up-and-out call option on a non-dividend paying stock where $S_0 = 50$, $K = 50$, the barrier is 60, $r = 10\%$, and $\sigma = 30\%$
- ▶ Any boundary can be chosen but the natural one is
$$c(S, 0.75) = \text{MAX}(S - 50, 0) \text{ when } S < 60$$
$$c(60, t) = 0 \text{ when } 0 \leq t \leq 0.75$$

EXAMPLE (CONTINUED)

We might try to match the following points on the boundary

$$c(S, 0.75) = \text{MAX}(S - 50, 0) \text{ for } S < 60$$

$$c(60, 0.50) = 0$$

$$c(60, 0.25) = 0$$

$$c(60, 0.00) = 0$$

EXAMPLE CONTINUED

(SEE TABLE 24.1, PAGE 572)

We can do this as follows:

+1.00 call with maturity 0.75 & strike 50

-2.66 call with maturity 0.75 & strike 60

+0.97 call with maturity 0.50 & strike 60

+0.28 call with maturity 0.25 & strike 60

EXAMPLE (CONTINUED)

- ▶ This portfolio is worth 0.73 at time zero compared with 0.31 for the up-and out option
- ▶ As we use more options the value of the replicating portfolio converges to the value of the exotic option
- ▶ For example, with 18 points matched on the horizontal boundary the value of the replicating portfolio reduces to 0.38; with 100 points being matched it reduces to 0.32

USING STATIC OPTIONS REPLICATION

- ▶ To hedge an exotic option we short the portfolio that replicates the boundary conditions
- ▶ The portfolio must be unwound when any part of the boundary is reached

CHAPTER 22 (MACDONALD)

Exotic Options: II

OUTLINE

- ▶ Simple options that are used to build more complex ones
 - ▶ Simple all-or-nothing options
 - ▶ All-or-nothing barrier options

- ▶ Multivariate options
 - ▶ Quantos (equity-linked forwards)
 - ▶ Currency-linked options
 - ▶ Rainbow options

- ▶ Throughout, assume that there are two processes

$$\frac{dS}{S} = (\alpha - \delta)dt + \sigma dZ$$

$$\frac{dQ}{Q} = (\alpha_Q - \delta_Q)dt + \sigma_Q dZ_Q$$

and that the correlation between dZ and dZ_Q is ρdt

NOMENCLATURE

TABLE 22.1

Option nomenclature used in this chapter.

Notation	Meaning
<i>Asset</i>	Payment at expiration is one unit of the asset
<i>Cash</i>	Payment at expiration is \$1
<i>Call</i>	Payment received if $S_T > K$
<i>Put</i>	Payment received if $S_T < K$
<i>UI</i>	Up and in: Payment received only if barrier $H > S_0$ is hit
<i>DI</i>	Down and in: Payment received only if barrier $H < S_0$ is hit
<i>UO</i>	Up and out: Payment received only if barrier $H > S_0$ is not hit
<i>DO</i>	Down and out: Payment received only if barrier $H < S_0$ is not hit
<i>UR</i>	Up rebate: Rebate received at the time the barrier, $H > S_0$, is hit
<i>DR</i>	Down rebate: Rebate received at the time the barrier, $H < S_0$, is hit
<i>URDeferred</i>	Same as UR, except \$1 paid at expiration
<i>DRDeferred</i>	Same as DR, except \$1 paid at expiration

DEFINITIONS

TABLE 22.2

Definitions of expressions used in pricing formulas in this chapter.

$$d_1 = [\ln(S_t/K) + (r - \delta + 0.5\sigma^2)(T - t)]/\sigma\sqrt{T - t}$$

$$d_2 = d_1 - \sigma\sqrt{T - t}$$

$$d_3 = [\ln(H^2/S_t K) + (r - \delta + 0.5\sigma^2)(T - t)]/\sigma\sqrt{T - t}$$

$$d_4 = d_3 - \sigma\sqrt{T - t}$$

$$d_5 = [\ln(S_t/H) + (r - \delta + 0.5\sigma^2)(T - t)]/\sigma\sqrt{T - t}$$

$$d_6 = d_5 - \sigma\sqrt{T - t}$$

$$d_7 = [\ln(H/S_t) + (r - \delta + 0.5\sigma^2)(T - t)]/\sigma\sqrt{T - t}$$

$$d_8 = d_7 - \sigma\sqrt{T - t}$$

ALL-OR-NOTHING OPTIONS

- ▶ Simple all-or-nothing options pay the holder a discrete amount of cash or a share if some particular event occurs

- ▶ Cash-or-nothing

- ▶ Call: pays x if $S_T > K$ and zero otherwise

$$\text{CashCall}(S, K, \sigma, r, T - t, \delta) = xe^{-r(T-t)} N(d_2)$$

- ▶ Put: pays x if $S_T < K$ and zero otherwise

$$\text{CashPut}(S, K, \sigma, r, T - t, \delta) = xe^{-r(T-t)} N(-d_2)$$

- ▶ Asset-or-nothing

- ▶ Call: pays S (one unit share) if $S_T > K$ and zero otherwise

$$\text{AssetCall}(S, K, \sigma, r, T - t, \delta) = e^{-r(T-t)} SN(d_1)$$

- ▶ Put: pays S (one unit share) if $S_T < K$ and zero otherwise

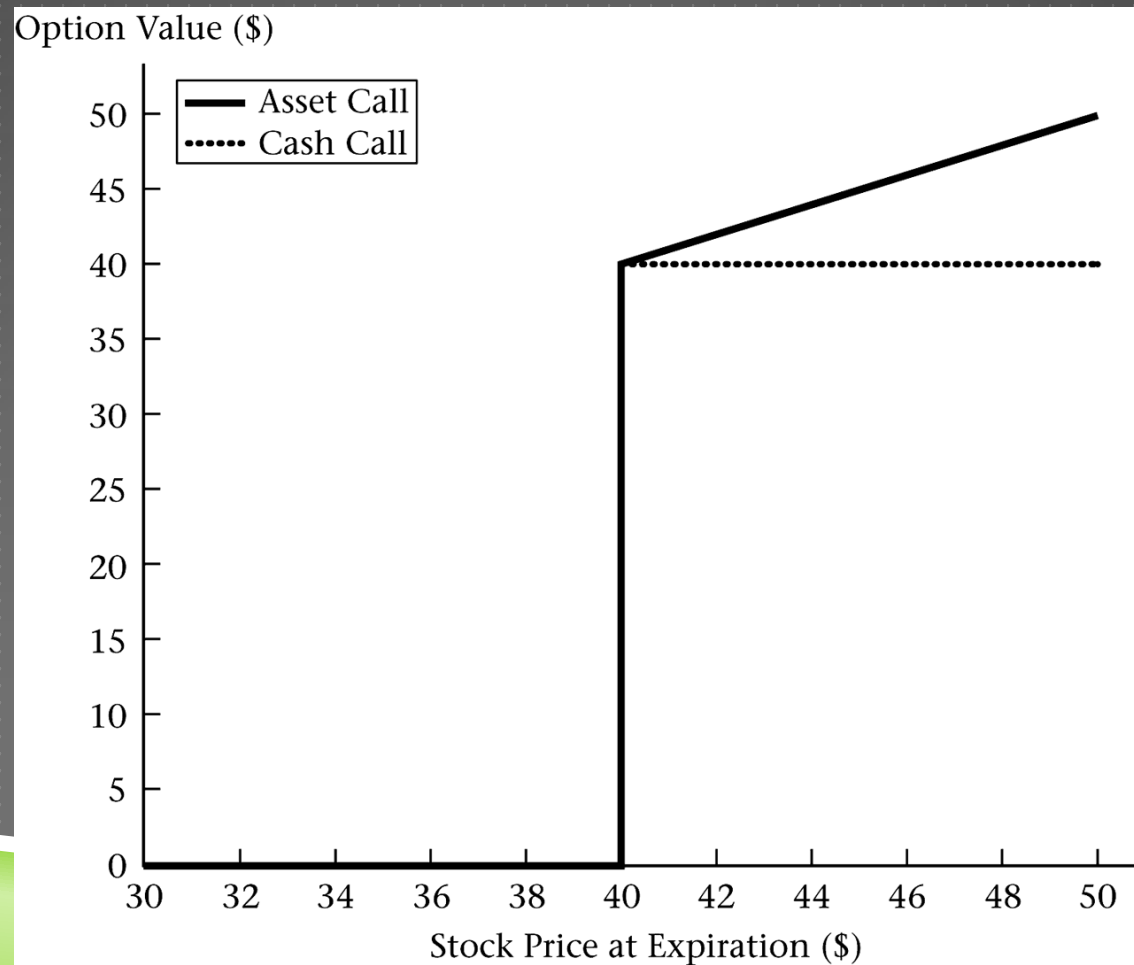
$$\text{AssetPut}(S, K, \sigma, r, T - t, \delta) = e^{-r(T-t)} SN(-d_1)$$

ALL-OR-NOTHING OPTIONS (CONT'D)

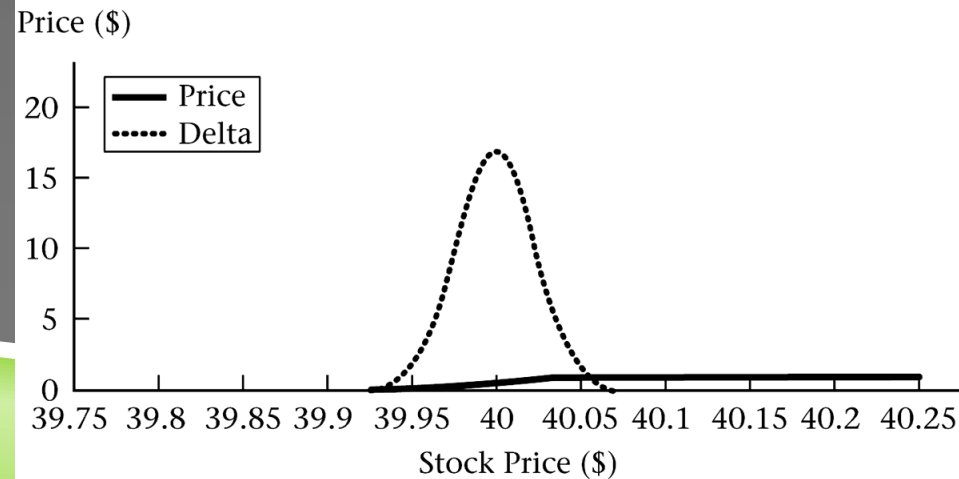
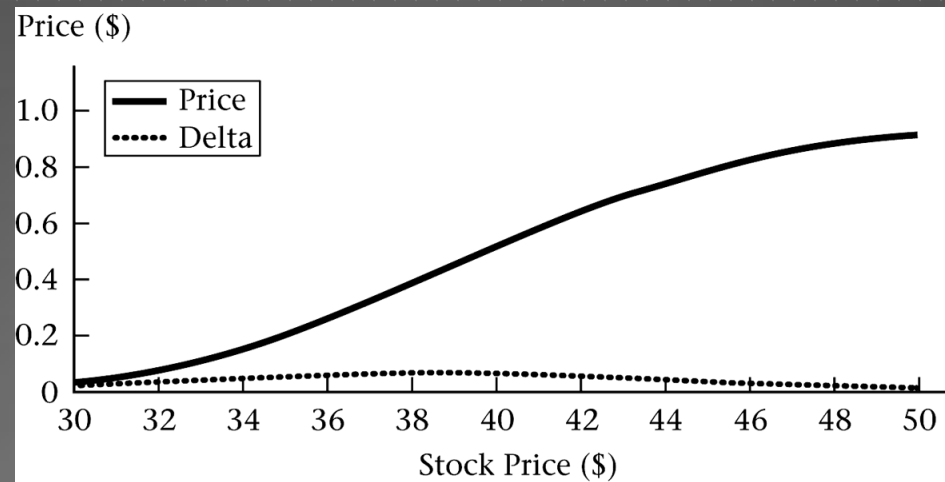
- ▶ + 1 asset-or-nothing call option with strike price K
– K cash-or-nothing call option with strike price K
= 1 ordinary call option with strike price K
- ▶ Similarly, a put option can be created by buying K cash-or-nothing puts, and buying 1 asset-or-nothing put
- ▶ A gap option that pays $S - K_1$ if $S - K_2$ can be created by buying an asset call and selling K_1 cash calls, both with the strike price K_2

$$\text{AssetCall}(S, K_2, \sigma, r, T - t, \delta) - K_1 \times \text{CashCall}(S, K_2, \sigma, r, T - t, \delta)$$

ALL-OR-NOTHING OPTIONS (CONT'D)



ALL-OR-NOTHING OPTIONS (CONT'D)



ALL-OR-NOTHING BARRIER OPTIONS

- ▶ Cash-or-nothing barrier option pays \$1 contingent on a barrier having or having not been reached
- ▶ Asset-or-nothing barrier option pays a share of stock worth S contingent on a barrier having or having not been reached
- ▶ Both (2) of the above can be calls or puts (2), knock-in or knock-out (2), and barrier maybe above or below (2) the price: $2^4=16$ varieties of basic all-or-nothing options exist
- ▶ Rebate option pays \$1 at the time and if the barrier is reached. Deferred rebate option pays at expiration

ALL-OR-NOTHING BARRIER OPTIONS (CONT'D)

- ▶ Down-and-in cash call with a barrier H : Pays \$1 if $S_t > K$ and barrier is hit from above during the life of the option
- ▶ The valuation formula for this option is reached by discounting the risk-neutral probability of this event

$$\text{CashDICall}(S, K, \sigma, r, T - t, \delta, H) =$$

$$e^{-r(T-t)} \left(\frac{H}{S} \right)^{2\frac{r-\delta}{\sigma^2}-1} N(d_4) \quad H \leq K$$

▶ Many other all-or-nothing barrier options can be valued using this result

$$e^{-r(T-t)} \left[N(d_2) - N(d_6) + \left(\frac{H}{S} \right)^{2\frac{r-\delta}{\sigma^2}-1} N(d_8) \right] \quad H > K$$

ALL-OR-NOTHING BARRIER OPTIONS (CONT'D)

- ▶ Deferred down rebate option pays \$I at expiration as long as the barrier has been hit from above during the option life
- ▶ This option is equivalent to a down-and-in cash call with a strike price $K = 0$. That is, the requirement of $S_t > K$ does not exist; it only requires the barrier to be hit

$$DRDeferred(S, \sigma, r, T - t, \delta, H) = CashDICall(S, 0, \sigma, r, T - t, \delta, H)$$

- ▶ Down-and-out cash call: we can create a synthetic cash call by buying down-and-in and down-and-out cash calls with the same barrier

$$CashDOCall(S, K, \sigma, r, T - t, \delta, H) = \\ CashCall(S, K, \sigma, r, T - t, \delta) - CashDICall(S, K, \sigma, r, T - t, \delta, H)$$

ALL-OR-NOTHING BARRIER OPTIONS (CONT'D)

- ▶ Down-and-in cash put: if you buy *both* a down-and-in cash call and put, you receive \$1 as long as the barrier is hit, which is the same payoff as a deferred rebate. Therefore

$$\begin{aligned} \text{CashDIPut}(S, K, \sigma, r, T - t, \delta, H) = \\ \text{DRDeferred}(S, \sigma, r, T - t, \delta, H) - \text{CashDICall}(S, K, \sigma, r, T - t, \delta, H) \end{aligned}$$

- ▶ Down-and-out cash put: buying down-and-in and down-and-out cash put creates an ordinary cash put. Therefore

$$\begin{aligned} \text{CashDOPut}(S, K, \sigma, r, T - t, \delta, H) = \\ \text{CashPut}(S, \sigma, r, T - t, \delta) - \text{CashDIPut}(S, K, \sigma, r, T - t, \delta, H) \end{aligned}$$

ALL-OR-NOTHING BARRIER OPTIONS (CONT'D)

- ▶ Up-and-in cash put: the valuation of this is similar to a down-and-in cash call

$$\text{CashUIPut}(S, K, \sigma, r, T - t, \delta, H) =$$

$$e^{-r(T-t)} \left(\frac{H}{S} \right)^{2 \frac{r-\delta}{\sigma^2} - 1} N(-d_4) \quad H \geq K$$
$$e^{-r(T-t)} \left[N(-d_2) - N(-d_6) + \left(\frac{H}{S} \right)^{2 \frac{r-\delta}{\sigma^2} - 1} N(-d_8) \right] \quad H < K$$

ALL-OR-NOTHING BARRIER OPTIONS (CONT'D)

- ▶ Deferred up rebate: similar to valuing the deferred down rebate, this time we set $K=\infty$, to obtain a claim paying \$I at expiration as long as the barrier is reached

$$URDeferred(S, \sigma, r, T - t, \delta, H) = CashUIPut(S, \infty, \sigma, r, T - t, \delta, H)$$

- ▶ Up-and-out cash put

$$CashUOPut(S, K, \sigma, r, T - t, \delta, H) = \\ CashPut(S, K, \sigma, r, T - t, \delta) - CashUIPut(S, K, \sigma, r, T - t, \delta, H)$$

ALL-OR-NOTHING BARRIER OPTIONS (CONT'D)

- ▶ Up-and-out cash call

$$\text{CashUICall}(S, K, \sigma, r, T - t, \delta, H) =$$

- ▶ Up-and-in cash call $\text{URDeferred}(S, K, \sigma, r, T - t, \delta) - \text{CashUIPut}(S, K, \sigma, r, T - t, \delta, H)$

$$\text{CashUOCall}(S, K, \sigma, r, T - t, \delta, H) =$$

$$\text{CashCall}(S, K, \sigma, r, T - t, \delta) - \text{CashUICall}(S, K, \sigma, r, T - t, \delta, H)$$

ALL-OR-NOTHING BARRIER OPTIONS (CONT'D)

- ▶ Asset-or-nothing barrier options: one can view these options as cash-or-nothing options denominated in cash-or-nothing options shares rather than cash
- ▶ Therefore, using the change of numeraire transformation from Chapter 21, we can obtain the pricing formulas
- ▶ Replace δ by $\delta - \sigma^2$, and multiply the cash-or-nothing formula with by $S_0 e^{(r - \delta)(T - t)}$

$$\text{AssetDICall}(S, K, \sigma, r, T - t, \delta, H)$$

- ▶ The remaining seven asset-or-nothing formulas can be created in the same way

$$= S e^{(r - \delta)(T - t)} \text{CashDICall}(S, K, \sigma, r, T - t, \delta - \sigma^2, H)$$

REBATE OPTIONS

▶ Down rebate options $DR(S, \sigma, r, T - t, \delta, H) = \left(\frac{H}{S}\right)^{h_1} N(Z_1) + \left(\frac{H}{S}\right)^{h_2} N(Z_2)$

▶ Up rebate options $UR(S, \sigma, r, T - t, \delta, H) = \left(\frac{H}{S}\right)^{h_1} N(-Z_1) + \left(\frac{H}{S}\right)^{h_2} N(-Z_2)$

where, letting $g = \sqrt{\left(r - \delta - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}$ then

$$Z_1 = \left[\ln(H / S) + g(T - t) \right] / \sigma \sqrt{T - t} \quad Z_2 = \left[\ln(H / S) + g(T - t) \right] / \sigma \sqrt{T - t}$$

$$h_1 = \left(\frac{r - \delta}{\sigma^2} - \frac{1}{2} \right) + \sqrt{\left(\frac{r - \delta}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}} \quad h_2 = \left(\frac{r - \delta}{\sigma^2} - \frac{1}{2} \right) + \sqrt{\left(\frac{r - \delta}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}}$$

BARRIER OPTIONS

- ▶ Ordinary barrier options can be valued using the all-or-nothing barrier options
 - ▶ Down-and-out call

$$\begin{aligned} &CallDownOut(S, K, \sigma, r, T - t, \delta, H) = \\ &\quad \text{Up-and-in put} \\ &AssetDOCalls(S, K, \sigma, r, T - t, \delta, H) - K \times CashDOCall(S, K, \sigma, r, T - t, \delta, H) \end{aligned}$$

$$\begin{aligned} &PutUpIn(S, K, \sigma, r, T - t, \delta, H) = \\ &K \times CashUIPut(S, K, \sigma, r, T - t, \delta, H) - AssetUIPut(S, K, \sigma, r, T - t, \delta, H) \end{aligned}$$

BARRIER OPTIONS (CONT'D)

- ▶ Capped options have the payoff of bull spreads except that the option is exercised the first time the stock price reaches the upper strike price
 - ▶ Example: Consider an option with a strike price of \$100 and a cap of \$120. When the stock price hits \$120 the option pays \$20. If the option expires without hitting \$120, it pays off $\max(S_T - 100, 0)$
- ▶ This option can be priced as the sum of
 - ▶ A rebate call that pays \$20 if the stock hits \$120 before expiration
 - ▶ A knock-out call with a strike of \$100, which knocks out at \$120

QUANTOS

- ▶ A quanto is a contract that allows an investor in one currency to hold an asset denominated in another currency without exchange rate risk
- ▶ It is also a derivative with a payoff that depends on the product or ratio of two prices
- ▶ Example: Nikkei put warrants that traded on the American Stock Exchange

CURRENCY-LINKED OPTIONS

- ▶ Foreign equity call struck in foreign currency
 - ▶ Buy an option completely denominated in foreign currency
 - ▶ Price it by using the BS formula with inputs appropriate for the asset being denominated in a different currency
 - ▶ Convert the price to dollar using the current exchange rate
- ▶ Foreign equity call struck in domestic currency
 - ▶ Consider a call option to buy Nikkei for the dollar-denominated strike price K . If the option is exercised K dollars is paid to acquire the Nikkei, which is worth $x_T Q_T$.
 - ▶ At expiration, the option is worth $V(x_T Q_T, T) = \max(0, x_T Q_T - K)$

CURRENCY-LINKED OPTIONS (CONT'D)

▶ Foreign equity call struck in domestic currency (cont'd)

▶ The volatility of $x_T Q_T$ is $v = \sqrt{\sigma_Q^2 + s^2 + 2\rho\sigma_Q s}$

▶ Using this volatility and the prepaid forward prices we have

$$C = x_0 Q_0 e^{-\delta T} N(d_1) - e^{-rt} K N(d_2)$$

$$d_1 = \frac{\ln(x_0 Q_0 e^{-\delta T} / e^{-rt} K) + 0.5v^2 t}{v\sqrt{t}}$$

$$d_2 = d_1 - v\sqrt{t}$$

CURRENCY-LINKED OPTIONS (CONT'D)

- ▶ Fixed exchange rate foreign equity call
 - ▶ Consider a call option denominated in the foreign currency, but with the option proceeds to be repatriated at a predetermined exchange rate \bar{x} . The payoff to this option with strike price K_f (denominated in the foreign currency) is

$$V(Q_T, T) = \bar{x} \times \max(0, Q_T - K_f) = \max(0, \bar{x}Q_T - \bar{x}K_f)$$

$$C = F_{0,T}^P(Q)N(d_1) - e^{-rt} K_f N(d_2)$$

$$d_1 = \frac{\ln(F_{0,t}^P(Q) / e^{-rt} K_f) + 0.5\sigma_Q^2 T}{\sigma_Q \sqrt{t}}$$

$$d_2 = d_1 - \sigma_Q \sqrt{t}$$

CURRENCY-LINKED OPTIONS (CONT'D)

- ▶ Equity-linked foreign exchange call
 - ▶ Consider an option that guarantees a minimum exchange rate K for the necessary quantity of currency to be exchanged when we convert the asset value back to the domestic currency. The payoff to such an insured position would be

$$\begin{aligned}
 & Q_T x_T + Q_T \max(0, K - x_T) \\
 = & Q_T K + Q_T \max(0, x_T - K) = Q_T K + \max(0, Q_T x_T - Q_T K)
 \end{aligned}$$

$$\begin{aligned}
 C &= Q_0 e^{(r_f - \delta_e - \rho\sigma s)T} \left[x_0 e^{-(r_f - \rho\sigma s)T} N(d_1) - e^{-rt} KN(d_2) \right] \\
 &= x_0 Q_0 e^{-\delta_e T} N(d_1) - K Q_0 e^{-(r + \delta_e + \rho\sigma s - r_f)T} N(d_2)
 \end{aligned}$$

$$d_1 = \frac{\ln(x_0 / K) + (r - r_f + \rho\sigma s + 0.5s^2)T}{s\sqrt{t}} \qquad d_2 = d_1 - s\sqrt{t}$$

OTHER MULTIVARIATE OPTIONS

- ▶ Exchange options in which the strike price is the price of a risky asset can be priced with a change of numeraire

- ▶ At maturity the exchange option with price $V(S_t, Q_t, t)$ pays

$$V(S_T, Q_T, T) = \max(S_T - Q_T, 0) = Q_T \times \max(S_T / Q_T - 1, 0)$$

- ▶ The formula for the value of the exchange option is

$$V(S, Q, t) = S e^{-\delta(T-t)} N(\hat{d}_1) - Q e^{-\delta_Q(T-t)} N(\hat{d}_2)$$

$$\hat{d}_1 = \frac{\ln(S / Q) + (\delta_Q - \delta + 0.5\hat{\sigma}^2)(T - t)}{\hat{\sigma}\sqrt{T - t}}$$

$$\hat{d}_2 = \hat{d}_1 - \hat{\sigma}\sqrt{T - t}$$

OTHER MULTIVARIATE OPTIONS (CONT'D)

- ▶ Rainbow options provide the greater of two assets and cash return: $\max(K, S_T, Q_T)$

$$\begin{aligned}
 & \text{RainbowCall}(S, Q, K, \sigma, s, \rho, \delta, \delta_Q, T - t) \\
 &= Se^{-\delta(T-t)} \left\{ N(d_{SQ}) - NN[-d_1(S), d_{SQ}, (\rho\sigma_Q - \sigma) / \hat{\sigma}] \right\} \\
 &+ Qe^{-\delta_Q(T-t)} \left\{ N(d_{QS}) - NN[-d_1(S), d_{QS}, (\rho\sigma - \sigma_Q) / \hat{\sigma}] \right\}
 \end{aligned}$$

- ▶ Basket options have payoffs that depend on the average of two or more asset prices

$$\max[0, S_{S\&P} - 0.5 \times (S_{Nikkei} + S_{Dax})]$$

- ▶ Basket options are valued using Monte Carlo simulation