MODELS OF THE BEHAVIOR OF STOCK PRICES

CATEGORIZATION OF STOCHASTIC PROCESSES

- Discrete time; discrete variable
- Discrete time; continuous variable
- Continuous time; discrete variable
- Continuous time; continuous variable

MODELING STOCK PRICES

- We can use any of the four types of stochastic processes to model stock prices
- The continuous time, continuous variable process proves to be the most useful for the purposes of valuing derivatives

MARKOV PROCESSES

- In a Markov process future movements in a variable depend only on where we are, not the history of how we got where we are
- We assume that stock prices follow Markov processes

WEAK-FORM MARKET EFFICIENCY

- This asserts that it is impossible to produce consistently superior returns with a trading rule based on the past history of stock prices. In other words technical analysis does not work.
- A Markov process for stock prices is clearly consistent with weak-form market efficiency

EXAMPLE OF A DISCRETE TIME CONTINUOUS VARIABLE MODEL

- A stock price is currently at $40
- At the end of 1 year it is considered that it will have a probability distribution of $\phi(40,10)$ where $\phi(\mu,\sigma)$ is a normal distribution with mean μ and standard deviation σ.

QUESTIONS

- What is the probability distribution of the stock price at the end of 2 years?
- ½ years?
- ¼ years?
- δt years?

Taking limits we have defined a continuous variable, continuous time process

VARIANCES & STANDARD DEVIATIONS

- In Markov processes changes in successive periods of time are independent
- This means that variances are additive
- Standard deviations are not additive

VARIANCES & STANDARD DEVIATIONS (CONTINUED)

- In our example it is correct to say that the variance is 100 per year.
- It is strictly speaking not correct to say that the standard deviation is 10 per year.

A WIENER PROCESS

- We consider a variable z whose value changes continuously
- The change in a small interval of time δt is δz
- The variable follows a Wiener process if
  - $\delta z = \varepsilon\sqrt{\delta t}$ where ε is a random drawing from $\phi(0,1)$
  - The values of δz for any 2 different (non-overlapping) periods of time are independent

PROPERTIES OF A WIENER PROCESS

- Mean of $[z(T) - z(0)]$ is 0
- Variance of $[z(T) - z(0)]$ is T
- Standard deviation of $[z(T) - z(0)]$ is $\sqrt{T}$

TAKING LIMITS . . .

- What does an expression involving dz and dt mean?
- It should be interpreted as meaning that the corresponding expression involving δz and δt is true in the limit as δt tends to zero
- In this respect, stochastic calculus is analogous to ordinary calculus

GENERALIZED WIENER PROCESSES

- A Wiener process has a drift rate (i.e. average change per unit time) of 0 and a variance rate of 1
- In a generalized Wiener process the drift rate and the variance rate can be set equal to any chosen constants

GENERALIZED WIENER PROCESSES (CONTINUED)

The variable x follows a generalized Wiener process with a drift rate of a and a variance rate of b^2 if

$$dx = a\,dt + b\,dz$$

GENERALIZED WIENER PROCESSES (CONTINUED)

$$\delta x = a\,\delta t + b\,\varepsilon\sqrt{\delta t}$$

- Mean change in x in time T is aT
- Variance of change in x in time T is b^2T
- Standard deviation of change in x in time T is

$$b\sqrt{T}$$

THE EXAMPLE REVISITED

- A stock price starts at 40 and has a probability distribution of $\phi(40,10)$ at the end of the year
- If we assume the stochastic process is Markov with no drift then the process is

$$dS = 10dz$$

- If the stock price were expected to grow by \$8 on average during the year, so that the year-end distribution is $\phi(48,10)$, the process is

$$dS = 8dt + 10dz$$

ITO PROCESS

- In an Ito process the drift rate and the variance rate are functions of time

$$dx=a(x,t)dt+b(x,t)dz$$

- The discrete time equivalent

  is only true in the limit as δt tends to zero

$$\delta x = a(x,t)\delta t + b(x,t)\varepsilon\sqrt{\delta t}$$

WHY A GENERALIZED WIENER PROCESS IS NOT APPROPRIATE FOR STOCKS

- For a stock price we can conjecture that its expected percentage change in a short period of time remains constant, not its expected absolute change in a short period of time
- We can also conjecture that our uncertainty as to the size of future stock price movements is proportional to the level of the stock price

AN ITO PROCESS FOR STOCK PRICES

$$dS = \mu S dt + \sigma S dz$$

where μ is the expected return σ is the volatility.

The discrete time equivalent is

$$\delta S = \mu S \delta t + \sigma S \varepsilon \sqrt{\delta t}$$

MONTE CARLO SIMULATION

- We can sample random paths for the stock price by sampling values for ε
- Suppose μ= 0.14, σ= 0.20, and δt = 0.01, then

$$\delta S = 0.0014S + 0.02S\varepsilon$$

MONTE CARLO SIMULATION – ONE PATH

Period	Stock Price at Start of Period	Random Sample for ε	Change in Stock Price, ΔS
0	20.000	0.52	0.236
1	20.236	1.44	0.611
2	20.847	-0.86	-0.329
3	20.518	1.46	0.628
4	21.146	-0.69	-0.262

ITO'S LEMMA

- If we know the stochastic process followed by x, Ito's lemma tells us the stochastic process followed by some function $G(x, t)$
- Since a derivative security is a function of the price of the underlying and time, Ito's lemma plays an important part in the analysis of derivative securities

TAYLOR SERIES EXPANSION

- A Taylor's series expansion of $G(x, t)$ gives

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + ½\frac{\partial^2 G}{\partial x^2}\delta x^2 + \frac{\partial^2 G}{\partial x \partial t}\delta x\,\delta t + ½\frac{\partial^2 G}{\partial t^2}\delta t^2 + \ldots$$

IGNORING TERMS OF HIGHER ORDER THAN ΔT

In ordinary calculus we have

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t$$

In stochastic calculus this becomes

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + ½\frac{\partial^2 G}{\partial x^2}\delta x^2$$

because δx has a component which is of order $\sqrt{\delta t}$

SUBSTITUTING FOR ΔX

Suppose

$$dx = a(x,t)dt + b(x,t)dz$$

so that

$$\delta x = a\,\delta t + b\,\varepsilon\sqrt{\delta t}$$

Then ignoring terms of higher order than δt

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + ½\frac{\partial^2 G}{\partial x^2}b^2\varepsilon^2\delta t$$

THE $\varepsilon^2\Delta T$ TERM

Since $\varepsilon \approx \phi(0,1)\ E(\varepsilon) = 0$

$$E(\varepsilon^2) - [E(\varepsilon)]^2 = 1$$

$$E(\varepsilon^2) = 1$$

It follows that $E(\varepsilon^2 \delta t) = \delta t$

The variance of δt is proportional to δt^2 and can be ignored. Hence

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2 \delta t$$

TAKING LIMITS

Taking limits $$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + ½ \frac{\partial^2 G}{\partial x^2} b^2 dt$$

Substituting $$dx = a\ dt + b\ dz$$

We obtain $$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\ dz$$

This is Ito's Lemma

APPLICATION OF ITO'S LEMMA TO A STOCK PRICE PROCESS

The stock price process is

$$dS = \mu S\, dt + \sigma S\, dz$$

For a function G of S and t

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S}\sigma S\, dz$$

EXAMPLES

1. The forward price of a stock for a contract maturing at time T

$$G = S\,e^{r(T-t)}$$

$$dG = (\mu - r)G\,dt + \sigma G\,dz$$

2. $G = \ln S$

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dz$$

THE BLACK-SCHOLES MODEL

THE STOCK PRICE ASSUMPTION

- Consider a stock whose price is S
- In a short period of time of length δt, the return on the stock is normally distributed:

$$\frac{\delta S}{S} \approx \phi\left(\mu \delta t, \sigma \sqrt{\delta t}\right)$$

where μ is expected return and σ is volatility

THE LOGNORMAL PROPERTY

- It follows from this assumption that

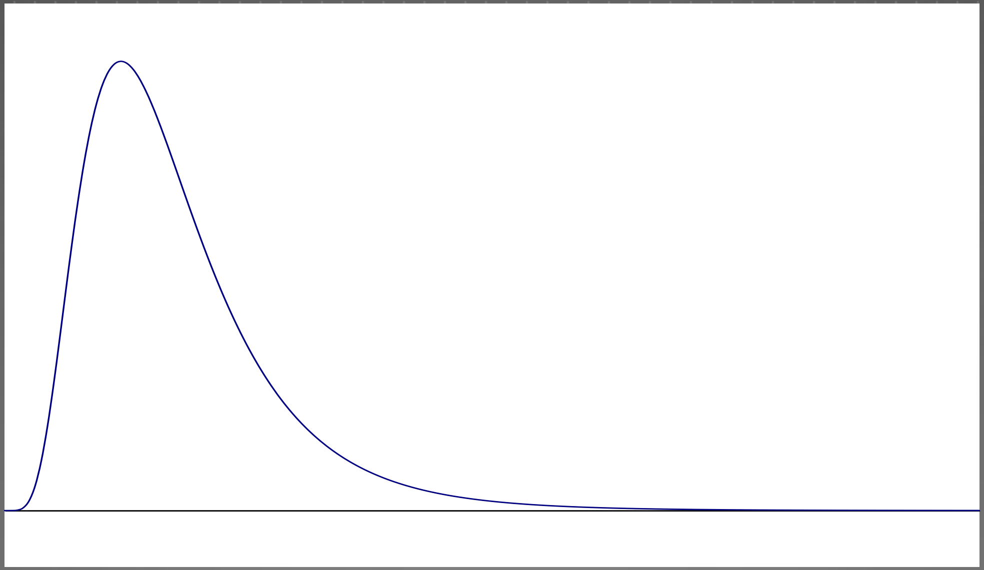
$$\ln S_T - \ln S_0 \approx \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right]$$

or

$$\ln S_T \approx \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right]$$

- Since the logarithm of S_T is normal, S_T is lognormally distributed

THE LOGNORMAL DISTRIBUTION



$$E(S_T) = S_0 \, e^{\mu T}$$

$$\text{var}(S_T) = S_0^{\,2} e^{2\mu T} (e^{\sigma^2 T} - 1)$$

CONTINUOUSLY COMPOUNDED RETURN, Η

$$S_T = S_0 \, e^{\eta T}$$

or

$$\eta = \frac{1}{T} \ln \frac{S_T}{S_0}$$

or

$$\eta \approx \phi\left(\mu - \frac{\sigma^2}{2}, \; \frac{\sigma}{\sqrt{T}} \right)$$

THE EXPECTED RETURN

- The expected value of the stock price is $S_0 e^{\mu T}$
- The expected return on the stock is
  $\mu - \sigma^2/2$ not μ

  This is because

  $$\ln[E(S_T / S_0)] \quad \text{and} \quad E[\ln(S_T / S_0)]$$

  are not the same

  $$E\left[\ln(S_T / S_0)\right] = \mu - \sigma^2 / 2$$
  $$\ln\left[E(S_T / S_0)\right] = \mu$$

μ AND $\mu-\sigma^2/2$

Suppose we have daily data for a period of several months

μ is the average of the returns in each day [$=E(\Delta S/S)$]

$\mu-\sigma^2/2$ is the expected return over the whole period covered by the data measured with continuous compounding (or daily compounding, which is almost the same)

THE VOLATILITY

- The volatility is the standard deviation of the continuously compounded rate of return in 1 year
- The standard deviation of the return in time Δt is

$$\sigma\sqrt{\Delta t}$$

- If a stock price is \$50 and its volatility is 25% per year what is the standard deviation of the price change in one day?

ESTIMATING VOLATILITY FROM HISTORICAL DATA

1. Take observations $S_0, S_1, \ldots, S_n$ at intervals of τ years
2. Calculate the continuously compounded return in each interval as:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

3. Calculate the standard deviation, s, of the u_i´s
4. The historical volatility estimate is: $\hat{\sigma} = \frac{s}{\sqrt{\tau}}$

NATURE OF VOLATILITY

- Volatility is usually much greater when the market is open (i.e. the asset is trading) than when it is closed
- For this reason time is usually measured in "trading days" not calendar days when options are valued

THE CONCEPTS UNDERLYING BLACK-SCHOLES

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes differential equation

THE DERIVATION OF THE BLACK-SCHOLES DIFFERENTIAL EQUATION

$$\delta S = \mu S\,\delta t + \sigma S\,\delta z$$

$$\delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + ½ \frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right)\delta t + \frac{\partial f}{\partial S}\sigma S\,\delta z$$

We set up a portfolio consisting of

-1: derivative

$+\dfrac{\partial f}{\partial S}$: shares

THE DERIVATION OF THE BLACK-SCHOLES DIFFERENTIAL EQUATION CONTINUED

The value of the portfolio Π is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time δt is given by

$$\delta\Pi = -\delta f + \frac{\partial f}{\partial S} \delta S$$

THE DERIVATION OF THE BLACK-SCHOLES DIFFERENTIAL EQUATION CONTINUED

The return on the portfolio must be the risk - free rate. Hence

$$\delta\Pi = r\,\Pi\delta t$$

We substitute for δf and δS in these equations to get the Black - Scholes differential equation :

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + ½\,\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r f$$

THE DIFFERENTIAL EQUATION

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- In a forward contract the boundary condition is

  $f = S - K$ when $t = T$

- The solution to the equation is

  $f = S - K e^{-r(T-t)}$

THE BLACK-SCHOLES FORMULAS
(SEE PAGES 295-297)

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

THE *N*(*X*) FUNCTION

- $N(x)$ is the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 is less than x
- See tables at the end of the book

PROPERTIES OF BLACK-SCHOLES FORMULA

- As S_0 becomes very large c tends to $S - Ke^{-rT}$ and p tends to zero
- As S_0 becomes very small c tends to zero and p tends to $Ke^{-rT} - S$

RISK-NEUTRAL VALUATION

- The variable μ does not appear in the Black-Scholes equation
- The equation is independent of all variables affected by risk preference
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- This leads to the principle of risk-neutral valuation

APPLYING RISK-NEUTRAL VALUATION

1. Assume that the expected return from the stock price is the risk-free rate
2. Calculate the expected payoff from the option
3. Discount at the risk-free rate

VALUING A FORWARD CONTRACT WITH RISK-NEUTRAL VALUATION

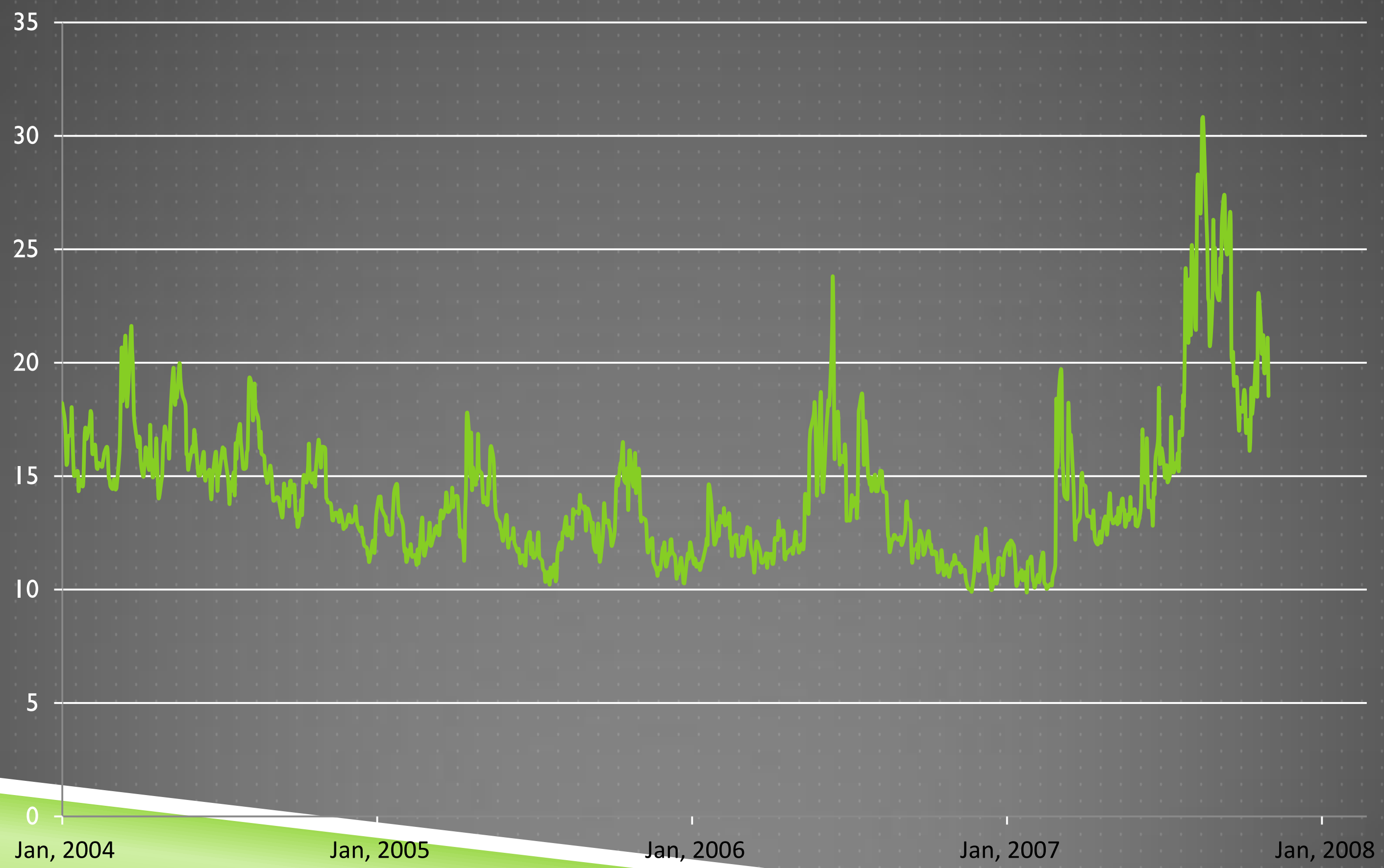
- Payoff is $S_T - K$
- Expected payoff in a risk-neutral world is $Se^{rT} - K$
- Present value of expected payoff is

$$e^{-rT}[Se^{rT} - K] = S - Ke^{-rT}$$

IMPLIED VOLATILITY

- The implied volatility of an option is the volatility for which the Black-Scholes price equals the market price
- The is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

THE VIX S&P500 VOLATILITY INDEX



CAUSES OF VOLATILITY

- Volatility is usually much greater when the market is open (i.e. the asset is trading) than when it is closed
- For this reason time is usually measured in "trading days" not calendar days when options are valued

DIVIDENDS

- European options on dividend-paying stocks are valued by substituting the stock price less the present value of dividends into Black-Scholes
- Only dividends with ex-dividend dates during life of option should be included
- The "dividend" should be the expected reduction in the stock price expected

THE GREEK LETTERS

EXAMPLE

- A bank has sold for $300,000 a European call option on 100,000 shares of a nondividend paying stock
- S_0 = 49, K = 50, r = 5%, σ = 20%, T = 20 weeks, μ = 13%
- The Black-Scholes value of the option is $240,000
- How does the bank hedge its risk to lock in a $60,000 profit?

NAKED & COVERED POSITIONS

- Naked position
  Take no action
- Covered position
  Buy 100,000 shares today

- Both strategies leave the bank exposed to significant risk

STOP-LOSS STRATEGY

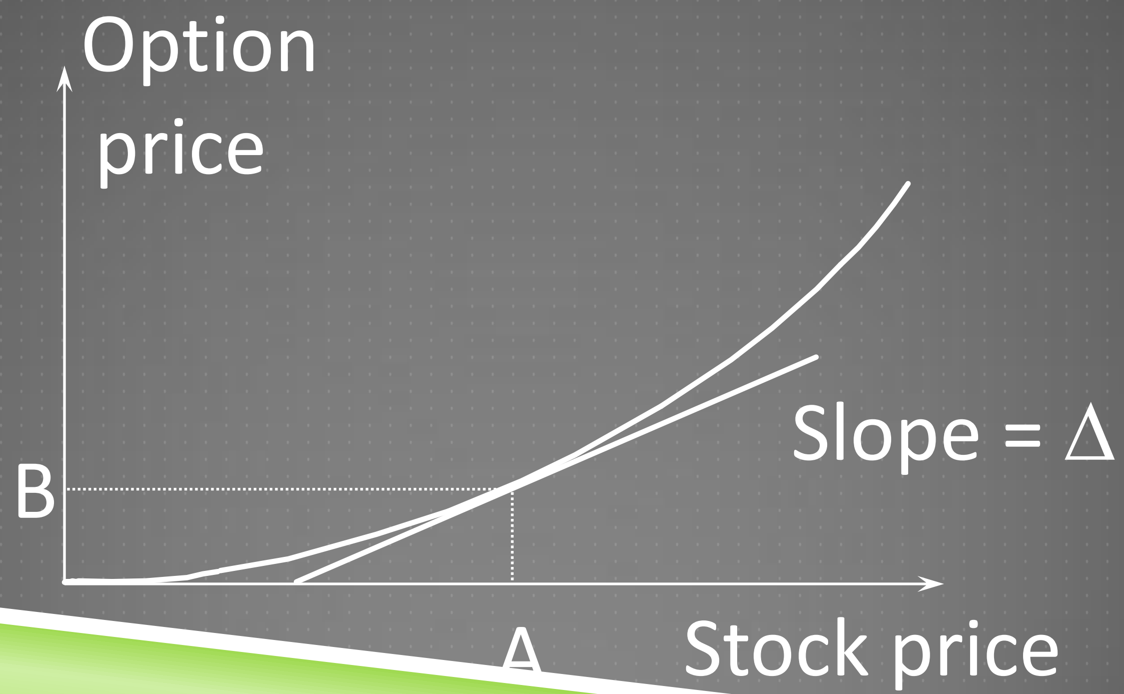
This involves:

- Buying 100,000 shares as soon as price reaches \$50
- Selling 100,000 shares as soon as price falls below \$50

This deceptively simple hedging strategy does not work well

DELTA

- Delta (Δ) is the rate of change of the option price with respect to the underlying



DELTA HEDGING

- This involves maintaining a delta neutral portfolio
- The delta of a European call on a stock paying dividends at rate q is $N(d_1)e^{-qT}$
- The delta of a European put is

$$e^{-qT}[N(d_1)-1]$$

DELTA HEDGING

- The hedge position must be frequently rebalanced
- Delta hedging a written option involves a "buy high, sell low" trading rule
- See Tables 15.2 (page 350) and 15.3 (page 351) for examples of delta hedging

USING FUTURES FOR DELTA HEDGING

- The delta of a futures contract is $e^{(r-q)T}$ times the delta of a spot contract
- The position required in futures for delta hedging is therefore $e^{-(r-q)T}$ times the position required in the corresponding spot contract

THETA

- Theta (Θ) of a derivative (or portfolio of derivatives) is the rate of change of the value with respect to the passage of time
- The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of the option declines

GAMMA

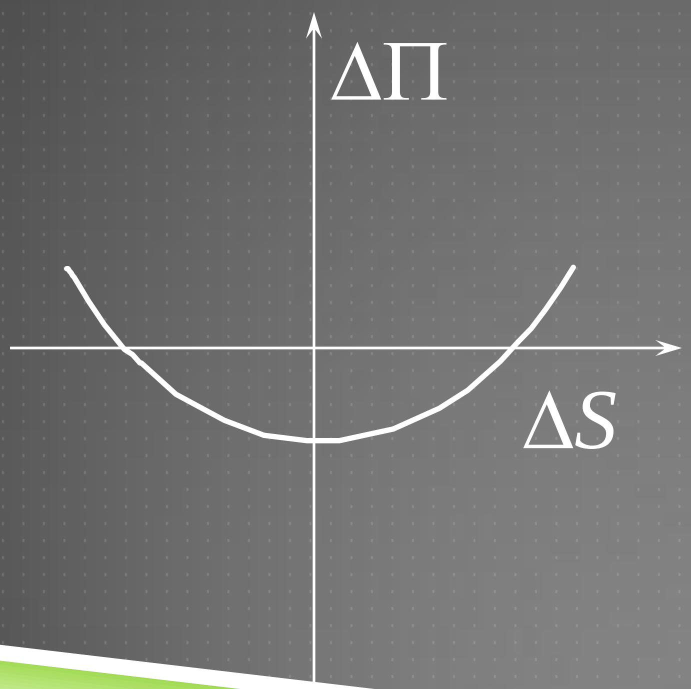
- Gamma (Γ) is the rate of change of delta (Δ) with respect to the price of the underlying asset
- Gamma is greatest for options that are close to the money (see Figure 15.9, page 358)

GAMMA ADDRESSES DELTA HEDGING ERRORS CAUSED BY CURVATURE

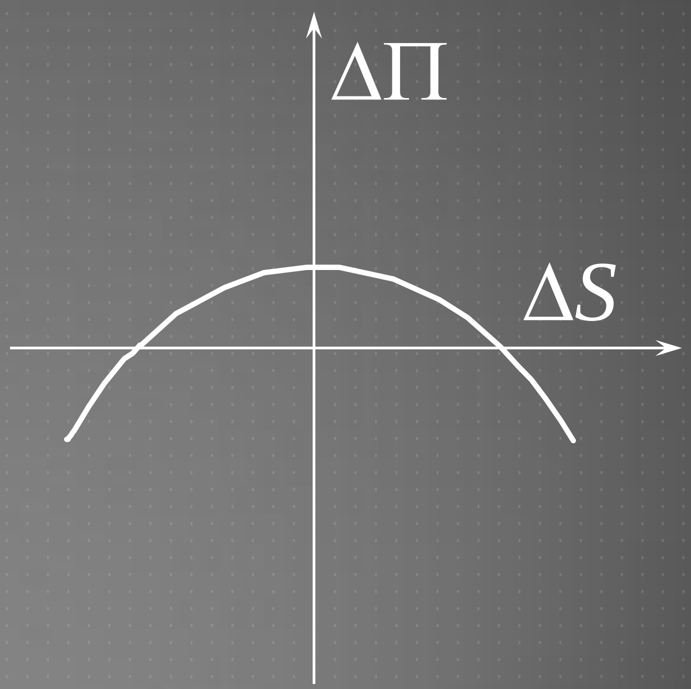
INTERPRETATION OF GAMMA

- For a delta neutral portfolio,

$$\Delta\Pi \approx \Theta\,\Delta t + \tfrac{1}{2}\Gamma\Delta S^{2}$$



Positive Gamma



Negative Gamma

RELATIONSHIP BETWEEN DELTA, GAMMA, AND THETA

For a portfolio of derivatives on a stock paying a continuous dividend yield at rate q

$$\Theta + (r - q)S\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

VEGA

- Vega ($\mathcal{V}$) is the rate of change of the value of a derivatives portfolio with respect to volatility
- Vega tends to be greatest for options that are close to the money (See Figure 15.11, page 361)

MANAGING DELTA, GAMMA, & VEGA

- Δ can be changed by taking a position in the underlying
- To adjust Γ & $\mathcal{V}$ it is necessary to take a position in an option or other derivative

RHO

- Rho is the rate of change of the value of a derivative with respect to the interest rate
- For currency options there are 2 rhos

HEDGING IN PRACTICE

- Traders usually ensure that their portfolios are delta-neutral at least once a day
- Whenever the opportunity arises, they improve gamma and vega
- As portfolio becomes larger hedging becomes less expensive

SCENARIO ANALYSIS

A scenario analysis involves testing the effect on the value of a portfolio of different assumptions concerning asset prices and their volatilities

HEDGING VS CREATION OF AN OPTION SYNTHETICALLY

- When we are hedging we take positions that offset $\Delta, \Gamma, \mathcal{V}$, etc.
- When we create an option synthetically we take positions that match Δ, Γ, & $\mathcal{V}$

PORTFOLIO INSURANCE

- In October of 1987 many portfolio managers attempted to create a put option on a portfolio synthetically
- This involves initially selling enough of the portfolio (or of index futures) to match the Δ of the put option

PORTFOLIO INSURANCE

- As the value of the portfolio increases, the Δ of the put becomes less negative and some of the original portfolio is repurchased
- As the value of the portfolio decreases, the Δ of the put becomes more negative and more of the portfolio must be sold

PORTFOLIO INSURANCE

The strategy did not work well on October 19, 1987...